Semidirect products in ideally exact categories

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Dedicated to Professor Hvedri Inassaridze on the occasion of his ninetieth birthday

Abstract

We extend the notion of semidirect product from *semi-abelian* to *ideally exact* context, and illustrate it by considering the case of unital commutative rings.

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1 Introduction

The categorical notion of semidirect product defined via monadicity of a certain kernel functor was first considered in [4], and then more systematically re-introduced in the context of semi-abelian categories (in the sense of [6]) in [1]. The purpose of the present paper is to suggest extending this context from semi-abelian to "ideally exact" just as it was done for the theory of ideals in [5]. As a simple motivation for such extension, one might ask, should not split epimorphisms of unital rings always be 'the same' as semidirect product projections? Note that although non-unital (=not-necessarily-unital) rings are more general than unital ones, describing the category of their split epimorphisms is a different rather than a more general problem. In the last section here we consider the even simpler case of commutative rings, aiming, however, not to obtain new results but only to illustrate what was done at the categorical level.

The theory of semidirect products is indeed related with the theory of ideals. In a semi-abelian category $\mathscr{X}$, every split epimorphism $\alpha : A \to B$ with a fixed section $\beta : B \to A$ determines an internal action ξ of B on the kernel X of α (in the sense of [1]) and a canonical isomorphism $B \ltimes (X, \xi) \to A$. More generally, in an ideally exact category $\mathscr{A}$, we do the same by using an adjunction $(F, U, \eta, \varepsilon) : \mathscr{X} \to \mathscr{A}$ with a semi-abelian $\mathscr{X}$ and taking $X \in \mathscr{X}$ to be the kernel of $U(\alpha)$. One can then think of X as an *ideal* of A as in the case of $\mathscr{A}$ being the category of unital rings and U being the forgetful functor 'from unital to non-unital'. Accordingly, our semidirect products depend on the choice of U and we call them *U-semidirect products*. There is, however, a *canonical choice* with $\mathscr{X} = (\mathscr{A} \downarrow 0)$, which is (not the same but) equivalent to what one might call the *best choice* in the case of unital rings, as mentioned at the very end of Section 3.

In order to express its main message most clearly, the paper is made very short; in particular we omitted: (a) the monoidal-categorical counterpart of what is done in [1] in the semi-abelian case; (b) the explicit presentation of semidirect product via the left adjoint of the comparison functor involved, which is also done in [1] in the semi-abelian case; and (c) some other straightforward calculations. However, the main monadicity theorem that allowed us to introduce semidirect products is proved in detail in Section 2.

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2 The monadicity theorem

Let $(F, U, \eta, \varepsilon) : \mathscr{X} \to \mathscr{A}$ an adjunction in which the category $\mathscr{A}$ has pushouts. For a fixed object B in $\mathscr{A}$ we construct the induced adjunction

$$(F', U', \eta', \varepsilon') : \mathrm{Pt}(U(B)) \to \mathrm{Pt}(B)$$

as follows:

- $\mathrm{Pt}(B)$ is the Bourn's *category of points* over B, that is, the category of triples (A, α, β) in which $\alpha : A \to B$ and $\beta : B \to A$ are morphisms in $\mathscr{A}$ with $\alpha\beta = 1_B$;
- similarly, $\mathrm{Pt}(U(B))$ is the category of triples (X, φ, ψ) in which $\varphi : X \to U(B)$ and $\psi : U(B) \to X$ are morphisms in $\mathscr{X}$ with $\varphi\psi = 1_{U(B)}$;
- $F' : \mathrm{Pt}(U(B)) \to \mathrm{Pt}(B)$ is defined by

  $$F'(X, \varphi, \psi) = (B +_{FU(B)} F(X), [1_B, \varepsilon_B F(\varphi)], \iota_1),$$

  using the pushout

  $$\begin{array}{ccc} B +_{FU(B)} F(X) & \xleftarrow{\iota_2} & F(X) \\ {\scriptstyle \iota_1}\uparrow & & \uparrow{\scriptstyle F(\psi)} \\ B & \xleftarrow[\varepsilon_B]{} & FU(B) \end{array}$$

- $U' : \mathrm{Pt}(B) \to \mathrm{Pt}(U(B))$ is defined by $U(A, \alpha, \beta) = (U(A), U(\alpha), U(\beta))$;
- $\eta'_{(X,\varphi,\psi)} : (X, \varphi, \psi) \to (U(B +_{FU(B)} F(X)), U([1_B, \varepsilon_B F(\varphi)]), U(\iota_1))$ is defined as the composite of

  $$X \xrightarrow{\eta_X} UF(X) \xrightarrow{U(\iota_2)} U(B +_{FU(B)} F(X));$$

- $\varepsilon'_{(A,\alpha,\beta)} = [\beta, \varepsilon_A] : (B +_{FU(B)} FU(A), [1_B, \varepsilon_B FU(\alpha)], \iota_1) \to (A, \alpha, \beta)$.

Next, assuming also that $\mathscr{X}$ is pointed and has finite coproducts, and in particular a zero object 0 (using the same symbol as for zero morphisms), let

$$(F'', U'', \eta'', \varepsilon'') : \mathscr{X} \to \mathrm{Pt}(U(B))$$

be the adjunction, considered in [4] and in several subsequent papers in the case of $U = 1_{\mathscr{X}}$; it has:

- $F'' : \mathscr{X} \to \mathrm{Pt}(U(B))$ defined by $F''(X) = (U(B) + X, [1, 0], \iota_1)$ (here and below we omit the index at 1);
- $U'' : \mathrm{Pt}(U(B)) \to \mathscr{X}$ defined by $U''(X, \varphi, \psi) = \mathrm{Ker}(\varphi)$;
- with the kernel of $[1, 0] : U(B) + X \to U(B)$ written as $U(B) \flat X = (U(B) \flat X, \kappa_{B,X})$, $\eta''_X : X \to U(B) \flat X$ defined via the commutative diagram

  $$\begin{array}{ccccc} X & & & & \\ {\scriptstyle \eta''_X}\downarrow & \searrow^{\iota_2} & & \searrow^{0} & \\ U(B) \flat X & \xrightarrow[\kappa_{B,X}]{} & U(B) + X & \xrightarrow[{[1,0]}]{} & U(B) \end{array}$$

- $\varepsilon_{(X,\varphi,\psi)} = [\psi, \ker(\varphi)] : (U(B) + \mathrm{Ker}(\varphi), [1,0], \iota_1) \to (X, \varphi, \psi)$.

Finally, we introduce the composite adjunction

$$(F^B, U^B, \eta^B, \varepsilon^B) = (F', U', \eta', \varepsilon')(F'', U'', \eta'', \varepsilon'') : \mathscr{X} \to \mathrm{Pt}(B),$$

and the purpose of this section is:

Theorem 2.1. Let $(F, U, \eta, \varepsilon) : \mathscr{X} \to \mathscr{A}$ be an adjunction satisfying the following conditions:

(a) the category $\mathscr{X}$ is semi-abelian in the sense of [6];

(b) the category $\mathscr{A}$ is Barr exact;

(c) the functor $U : \mathscr{A} \to \mathscr{X}$ reflects isomorphisms and preserves regular epimorphisms.

Then the category $\mathscr{A}$ is ideally exact in the sense of [5], and the functors U and U^B (for any object B in $\mathscr{A}$) are monadic.

Proof. We observe:

(i) Since $\mathscr{X}$ is semi-abelian and $U : \mathscr{A} \to \mathscr{X}$ reflects isomorphisms, $\mathscr{A}$ is Bourn protomodular by the result of Example 3 in Section 6 of [2].

(ii) This makes $\mathrm{Pt}(B)$ protomodular by the result of Example 4 in Section 6 of [2].

(iii) As follows from (i) and (ii), $\mathscr{A}$ and $\mathrm{Pt}(B)$ are Mal'tsev categories by Proposition 17 in [3].

(iv) Since U is a right adjoint that preserves regular epimorphisms and $\mathscr{A}$ is Barr exact, (iii) implies that U preserves reflexive coequalizers.

(v) Since U reflects isomorphisms, (iv) implies that U is monadic.

(vi) (v) implies that $\mathscr{A}$ is ideally exact by Theorem 3.1 and Definition 3.2 of [5].

(vii) (vi) implies that $\mathscr{A}$ has finite colimits (by Theorem 3.3 of [5]), and so the functors $F' : \mathrm{Pt}(U(B)) \to \mathrm{Pt}(B)$ and $F^B = F'F'' : \mathscr{X} \to \mathrm{Pt}(B)$ are well defined for each B in $\mathscr{A}$.

(viii) $U' : \mathrm{Pt}(B) \to \mathrm{Pt}(U(B))$ preserves reflexive coequalizers by (iv).

(ix) Similarly, $U'' : \mathrm{Pt}(U(B)) \to \mathscr{X}$ preserves coequalizers of refrexive relations (see Theorem 3.4(a) in [4]).

(x) Since U reflects isomorphisms, U' also does.

(xi) Since $\mathscr{X}$ is Bourn protomodular, U'' reflects isomorphisms.

As follows from (viii)-(xi), the functor $U^B = U''U'$ preserves coequalizers of refrexive relations and reflects isomorphisms, which makes it monadic. Q.E.D.

3 Dependent and canonical semidirect products

Let $(F, U, \eta, \varepsilon) : \mathscr{X} \to \mathscr{A}$ be a fixed adjunction satisfying conditions (a)-(c) of Theorem 2.1; in particular, the category $\mathscr{A}$ is ideally exact.

For an object B in $\mathscr{A}$, let $T^B = (T^B, \eta^B, \mu^B)$ be the monad on $\mathscr{X}$ corresponding to the adjunction

$$(F^B, U^B, \eta^B, \varepsilon^B) : \mathscr{X} \to \mathrm{Pt}(B),$$

let $\mathscr{X}^B$ be the category of T^B-algebras, and let

$$(\overline{F}^B, \overline{U}^B, \overline{\eta}^B, \overline{\varepsilon}^B) : \mathscr{X}^B \to \mathrm{Pt}(B),$$

be the comparison equivalence. We introduce:

Definition 3.1. *Given objects B in $\mathscr{A}$ and (X, ξ) in $\mathscr{X}^B$, we write*

$$\overline{F}^B(X, \xi) = (B \ltimes_U (X, \xi), \pi_{B,X,\xi}, \iota_{B,X,\xi})$$

and say that

$$B \ltimes_U (X, \xi) = (B \ltimes_U (X, \xi), \pi_{B,X,\xi}, \iota_{B,X,\xi})$$

*is the U-*semidirect product *of B and (X, ξ).*

Next, given an arbitrary ideally exact category $\mathscr{A}$, we will make a special choice for the adjunction $(F, U, \eta, \varepsilon) : \mathscr{X} \to \mathscr{A}$, in fact as in [5], where it is shown that all the conditions we need are satisfied. Writing 0 for an initial object in $\mathscr{A}$ (since there will be no confusion with a zero object in $\mathscr{X}$) and $!_A : 0 \to A$ for the unique morphism from 0 to any given object A in $\mathscr{A}$, we take:

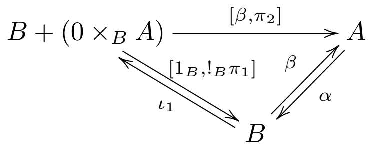
- $\mathscr{X} = (\mathscr{A} \downarrow 0)$;
- $F : (\mathscr{A} \downarrow 0) \to \mathscr{A}$ to be the forgetful functor (=the underlying object functor) defined by $F(X, \chi) = X$;
- $U : \mathscr{A} \to (\mathscr{A} \downarrow 0)$ defined by $U(A) = (0 \times A, \pi_1)$;
- $\eta_{(X,\chi)} = \langle \chi, 1_X \rangle : (X, \chi) \to (0 \times X, \pi_1)$;
- $\varepsilon_A = \pi_2 : 0 \times A \to A$.

A bit long but straightforward calculation shows that in this case the adjunction $(F^B, U^B, \eta^B, \varepsilon^B) : \mathscr{X} \to \mathrm{Pt}(B)$ has:

- $F^B : (\mathscr{A} \downarrow 0) \to \mathrm{Pt}(B)$ is defined by $F^B(X, \chi) = (B + X, [1_B, !_B\chi], \iota_1)$;
- $U^B : \mathrm{Pt}(B) \to (\mathscr{A} \downarrow 0)$ is defined by $U^B(A, \alpha, \beta) = (0 \times_B A, \pi_1)$;
- for an object (X, χ) in $(\mathscr{A} \downarrow 0)$, $\eta^B_{(X,\chi)} =$

$$\begin{array}{ccc} X & \xrightarrow{\langle \chi, \iota_2 \rangle} & 0 \times_B (B + X) \\ & {\scriptstyle \chi} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & 0 & \end{array}$$

- for an object (A, α, β) in $\mathrm{Pt}(B)$, $\varepsilon^B_{(A,\alpha,\beta)} =$



Definition and Remark 3.2. *For this special choice of the adjunction $(F, U, \eta, \varepsilon)$, it convenient to present the semidirect product of B and $((X, \chi), \xi)$ as*

$$B \ltimes ((X, \chi), \xi) = (B \ltimes ((X, \chi), \xi), \pi_{B,(X,\chi),\xi}, \iota_{B,(X,\chi),\xi})$$

and call it the canonical semidirect product *of B and $((X, \chi), \xi)$. In this case, the last assertion of Theorem 2.1 follows from Theorem 3.4 of [4], and the object $B \ltimes ((X, \chi), \xi)$ is the same as what would be written as $((X, \chi, !_X), \xi) \rtimes (B, !_B)$ in the notation of [4] (although $(X, \chi, !_X)$ was written as one letter there). Note also that, for a semi-abelian $\mathscr{A}$, the $1_{\mathscr{A}}$-semidirect products are the same as the semidirect products in the sense of [1] (which themselves are a special case of the semidirect products in the sense of [4]).*

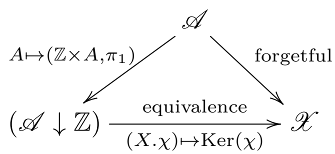
Remark 3.1. *Let $\mathscr{A}$ be the category of unital rings. In this 'classical' case, one might first of all think of the following three choices of the functor $U : \mathscr{A} \to \mathscr{X}$:*

(a) what we called the canonical choice, where $\mathscr{X} = (\mathscr{A} \downarrow \mathbb{Z})$ and U is defined by $U(A) = (\mathbb{Z} \times A, \pi_1)$ (we are using the fact that the ring $\mathbb{Z}$ of integers is the initial object in $\mathscr{A}$);

(b) what seems to be the best (or most natural) choice, where U is the forgetful functor to the category of non-unital rings;

(c) taking U to be the forgetful functor to the category of abelian groups, in order to have an abelian $\mathscr{X}$.

However, the choices (a) and (b) are equivalent in the sense that for $\mathscr{X}$ being the category of non-unital rings we have the commutative diagram



4 Semidirect products of commutative rings

In this section we choose the adjunction $(F, U, \eta, \varepsilon) : \mathscr{X} \to \mathscr{A}$ as follows:

- $\mathscr{X}$ is the category of non-unital commutative rings, where "non-unital" means that the existence of the element 1 is not required;
- $\mathscr{A}$ is the category of unital commutative rings, where the existence of 1 is required and the morphisms are required to preserve it;

- $F : \mathscr{X} \to \mathscr{A}$ is defined by $F(X) = \mathbb{Z} \ltimes X$ ("Dorroh extension"), where $\mathbb{Z}$ is the ring of integers, the underlying abelian group of $\mathbb{Z} \ltimes X$ is the same as of $\mathbb{Z} \times X$, and the multiplication of $\mathbb{Z} \ltimes X$ is defined by
$$(z, x)(z', x') = (zz', zx' + z'x + xx');$$
- $U : \mathscr{A} \to \mathscr{X}$ is the forgetful functor;
- $\eta_X : X \to \mathbb{Z} \ltimes X$ is defined by $\eta_X(x) = (0, x)$;
- $\varepsilon_A : \mathbb{Z} \ltimes X \to A$ is defined by $\varepsilon_A(z, a) = z \cdot 1 + a$, where 1 denotes the element 1 of A.

In this case the adjunction $(F^B, U^B, \eta^B, \varepsilon^B) : \mathscr{X} \to \mathrm{Pt}(B)$ has (having in mind that the initial object in $\mathscr{A}$ is $\mathbb{Z}$):

- $F^B : \mathscr{X} \to \mathrm{Pt}(B)$ defined by
$$F^B(X) = (B \otimes (\mathbb{Z} \ltimes X), \pi_X, \iota_1),$$
where $\pi_X : B \otimes (\mathbb{Z} \ltimes X) \to B$ is determined by $b \otimes (z, x) \mapsto zb$ and ι_1 is defined by $\iota_1(b) = b \otimes (1, 0)$;
- $U^B : \mathrm{Pt}(B) \to \mathscr{X}$ defined by $U^B(A, \alpha, \beta) = \mathrm{Ker}(\alpha)$, where "Ker" is understood ring-theoretically, that is $\mathrm{Ker}(\alpha) = \{a \in A \mid \alpha(a) = 0\}$;
- identifying $U^B F^B(X) = \mathrm{Ker}(\pi_X : B \otimes (\mathbb{Z} \ltimes X) \to B)$ with $B \otimes X$,
$$\eta^B_X : X \to B \otimes X$$
defined by $\eta^B_X(x) = 1 \otimes x$;
- $\varepsilon^B_{(A,\alpha,\beta)} : (B \otimes (\mathbb{Z} \ltimes \mathrm{Ker}(\alpha)), \pi_{\mathrm{Ker}(\alpha)}, \iota_1) \to (A, \alpha, \beta)$ determined by
$$b \otimes (z, a) \mapsto \beta(b)(z \cdot 1 + a) = z\beta(b) + \beta(b)a.$$

We obtain:

Theorem 4.1. The monad $T^B = (T^B, \eta^B, \mu^B)$ corresponding to the adjunction $(F, U, \eta, \varepsilon) : \mathscr{X} \to \mathscr{A}$ can be described as follows:

(a) $T^B(X) = B \otimes X$;

(b) $\eta^B_X : X \to B \otimes X$ is defined by $\eta_X(x) = (0, x)$;

(c) $\mu^B_X : B \otimes B \otimes X \to B \otimes X$ is defined by $\mu^B_X(b \otimes b' \otimes x) = bb' \otimes x$.

Proof. We already have (a) and (b), and it remains to prove (c). According to the description of ε^B above, we know that
$$\varepsilon^B_{F^B(X)} : (B \otimes (\mathbb{Z} \ltimes \mathrm{Ker}(\pi_X)), \pi_{\mathrm{Ker}(\pi_X)}, \iota_1) \to (B \otimes (\mathbb{Z} \ltimes X), \pi_X, \iota_1)$$
is determined by $b \otimes (z, k) \mapsto z\iota_1(b) + \iota_1(b)k$, where we assume $b \in B$, $z \in \mathbb{Z}$, and $k \in \mathrm{Ker}(\pi_X) \subseteq B \otimes (\mathbb{Z} \ltimes X)$. Now, identifying again

- $\mathrm{Ker}(\pi_X)$ with $B \otimes X$,
- similarly, $\mathrm{Ker}(\pi_{\mathrm{Ker}(\pi_X)})$ with $B \otimes \mathrm{Ker}(\pi_X)$,
- and then further, $\mathrm{Ker}(\pi_{\mathrm{Ker}(\pi_X)})$ with $B \otimes B \otimes X$,

we can say that the map $\mu_X^B = U^B(\varepsilon^B_{F^B(X)})$ is determined by making the diagram

$$\begin{array}{ccc} B \otimes (\mathbb{Z} \ltimes \mathrm{Ker}(\pi_X)) & \xrightarrow{\;b\otimes(z,k)\mapsto z\iota_1(b)+\iota_1(b)k\;} & B \otimes (\mathbb{Z} \ltimes X) \\ \uparrow {\scriptstyle b\otimes b'\otimes x\mapsto b\otimes(0,b'\otimes(0,x))} & & \uparrow {\scriptstyle b\otimes x\mapsto b\otimes(0,x)} \\ B \otimes B \otimes X & \xrightarrow[\mu_X^B]{} & B \otimes X \end{array}$$

commute. For the top arrow in this diagram, since $\iota_1(b) = b\otimes(1,0)$ (by the construction of pushouts in the category of unital commutative rings), we have

$$b \otimes (0, b' \otimes (0, x)) \mapsto 0(b \otimes (1, 0)) + (b \otimes (1, 0))(b' \otimes (0, x)) = bb' \otimes (0, x),$$

and so $\mu_X^B(b \otimes b' \otimes x) = bb' \otimes x$ holds indeed. Q.E.D.

Corollary 4.2. The category $\mathscr{X}^B$ is the same as the category of commutative B-algebras in the sense of ring theory, that is, the category of B-modules X equipped with a commutative (non-unital) ring structure with $(bx)x' = b(xx') = x(bx')$ for all $b \in B$ and $x, x' \in X$. Q.E.D.

Further, it is easy to check that the category equivalence

$$\mathrm{Pt}(B) \sim \mathscr{X}^B$$

is nothing but the standard equivalence between the category of *augmented unital* commutative B-algebras and the category of *non-unital* commutative B-algebras. Under this equivalence an augmented unital B-algebra corresponds to the kernel of its augmentation, while a non-unital B-algebra X corresponds to the U-semidirect product $(B \ltimes_U (X, \xi), \pi_{B,X,\xi}, \iota_{B,X,\xi})$ for which:

- the underlying abelian group of $B \ltimes_U (X, \xi)$ is the same as of $B \times X$, and the multiplication of $B \ltimes_U (X, \xi)$ is defined by

  $$(b, x)(b', x') = (bb', bx' + b'x + xx'),$$

  and so it is a natural generalization of $\mathbb{Z} \times X$ from rings to B-algebras;

- $\pi_{B,X,\xi} : B \ltimes_U (X, \xi) \to B$ is defined by $\pi_{B,X,\xi}(b, x) = b$;
- $\iota_{B,X,\xi} : B \to B \ltimes_U (X, \xi)$ is defined by $\iota_{B,X,\xi}(b) = (b, 0)$.

In particular, when $B = \mathbb{Z}$, B-algebras, unital or not, are the same as rings, unital or not, respectively, and $\mathbb{Z} \ltimes_U (X, \xi) = \mathbb{Z} \ltimes X$, where ξ disappears since it is uniquely determined.

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