Extensibility for derivations of abelian extensions of Leibniz n-algebras and Wells exact sequence

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Dedicated to Professor Hvedri Inassaridze on the occasion of his 90th birthday

Abstract

We study the extensibility problem for a pair of derivations associated with an abelian extension of Leibniz n-algebras, and derive an exact sequence of the Wells type.

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1 Introduction

Leibniz n-algebras were introduced in [9] as the non-skew symmetric version of Nambu algebras [10, 18] or Lie n-algebras [12]. For $n = 2$, the Leibniz 2-algebras are ordinary Leibniz algebras [16]. Thus, the concept of Leibniz n-algebra is a simultaneous generalization of Lie, Leibniz, and Lie n-algebras. In this generalizing framework, it is natural to ask for the extension of results in Leibniz or Lie algebras categories to the category of Leibniz n-algebras.

Based on Leibniz cohomology [17], the Quillen cohomology of a Leibniz n-algebra is computed via the explicit cochain complex in [9]. Further (co)homological investigations of Leibniz n-algebras are treated in [4, 5, 6]. In particular, in [5] we introduced crossed modules of Leibniz n-algebras and describe the second cohomology as the set of equivalence classes of crossed extensions. In [6], we provided the Hopf type formulas for the higher homology of Leibniz n-algebras, studied in [4]. All these investigations exploit the remarkable properties of the so-called Daletskii-Takhtajan's functor from the category of Leibniz n-algebras to the category of Leibniz algebras, which is further explored in our recent paper [7].

In the presented work, we aimed to study the extensibility problem for derivations associated with an abelian extension of Leibniz n-algebras. In the case of groups, such an extensibility problem of automorphisms goes back to Baer [1]. Important progress was achieved by Wells in extensions of abstract groups [21], where a map, later called a Wells map, is defined and an exact sequence, also called a Wells exact sequence, connecting various automorphism groups is constructed. The Wells map and the extensibility of a pair of automorphisms associated with an (abelian) extension were studied in the context of various algebraic structures [3, 11, 13, 14, 15, 19]. In parallel, the extensibility of derivations associated with an abelian extension of Lie algebras, associative algebras and algebras with bracket have been studied in [2], [20] and [8], respectively.

Organization

After the introductory Section 1, the paper is organized into two more sections. Section 2 recalls basic definitions on Leibniz n-algebras, construction of cohomology complex and description of the zero and the first cohomologies via derivations and abelien extensions, respectively. The main

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results are presented in Section 3. Namely, the extensibility problem for derivations of Leibniz n-algebras associated with an abelian extension is stated and investigated, the Wells map is constructed (Definition 3.5), which is used to determine the necessary and sufficient condition for the extensibility of a pair of derivations associated to a given abelian extension of Leibniz n-algebras (Theorem 3.7). The Wells exact sequence connecting various vector spaces of derivations is also obtained (Theorem 3.9).

Conventions and notation.

Throughout the paper we fix a ground field $\mathbb{K}$. All vector spaces and algebras are $\mathbb{K}$-vector spaces and $\mathbb{K}$-algebras, and linear maps are $\mathbb{K}$-linear maps as well. Hom and $\otimes$ denote $\mathrm{Hom}_{\mathbb{K}}$ and $\otimes_{\mathbb{K}}$, respectively. For the composition of two maps f and g, we write simply gf. For the identity map on a set X we use the notation id_X. For any equivalence relation on a set X we write $\mathrm{cl}(x)$ to denote the equivalence class of an element $x \in X$.

2 Preliminaries

2.1 Leibniz n-algebras

A *Leibniz n-algebra* [9] is a vector space $\mathcal{L}$ equipped with an n-linear bracket $[-,\cdots,-] : \mathcal{L}^{\times n} \to \mathcal{L}$ satisfying the following fundamental identity

$$[[x_1,\dots,x_n],y_1,\dots,y_{n-1}] = \sum_{1\le i\le n} [x_1,\dots,x_{i-1},[x_i,y_1,\dots,y_{n-1}],x_{i+1},\dots,x_n] \tag{2.1}$$

for all $x_1,\dots,x_n,y_2,\dots,y_n \in \mathcal{L}$.

A homomorphism of Leibniz n-algebras $\mathcal{L} \to \mathcal{L}'$ is a linear map preserving the n-ary bracket. The respective category of Leibniz n-algebras will be denoted by ${}_{\mathbf{n}}\mathbf{Lb}$.

For $n = 2$, the identity (2.1) is equivalent to the Leibniz identity

$$[[x,y],z] = [[x,z],y] + [x,[y,z]],$$

so a Leibniz 2-algebra is simply a Leibniz algebra [16] and it is a *Lie algebra* if the condition $[x,x] = 0$ is fulfilled for all $x \in \mathcal{L}$. In general, if the n-ary bracket is skew-symmetric, that is

$$[x_{\sigma(1)},\dots,x_{\sigma(n)}] = \mathrm{sgn}(\sigma)[x_1,\dots,x_n],$$

for all $\sigma \in S_n$, then $\mathcal{L}$ is *a Lie n-algebra or Filippov algebra* [12, 18]. Here S_n stands for the permutation group on n elements, and $\mathrm{sgn}(\sigma) \in \{-1,\ 1\}$ denotes the signature of σ.

To a given Leibniz algebra $\mathcal{L}$, one assigns the Leibniz n-algebra with the same underlying vector space $\mathcal{L}$ endowed with the n-ary bracket defined by

$$[x_1,x_2,\dots,x_n] = [x_1,[x_2,\dots,[x_{n-1},x_n]\cdots]]$$

(see [9]). This assignment is functorial and provides the so-called forgetful functor

$$\mathsf{Lb} \to {}_{\mathrm{n}}\mathsf{Lb}, \quad \mathcal{L} \mapsto \mathcal{L}.$$

Conversely, the so-called *Daletskii-Takhtajan's functor* [10]

$$\mathcal{D}_n : {}_{\mathrm{n}}\mathsf{Lb} \to \mathsf{Lb}$$

assigns to a Leibniz n-algebra $\mathcal{L}$ the Leibniz algebra $\mathcal{D}_n(\mathcal{L})$ with the underlying vector space $\mathcal{L}^{\otimes n-1}$ and with the bracket operation given by

$$[x_1 \otimes \cdots \otimes x_{n-1}, x'_1 \otimes \cdots \otimes x'_{n-1}] = \sum_{1 \leq i \leq n-1} x_1 \otimes \cdots \otimes [x_i, x'_1, \ldots, x'_{n-1}] \otimes \cdots \otimes x_{n-1} \ .$$

A subalgebra $\mathcal{L}'$ of a Leibniz n-algebra $\mathcal{L}$ is called an *n-sided ideal* if $[x_1, \ldots, x_n] \in \mathcal{L}'$ as soon as $x_i \in \mathcal{L}'$ for some i, $1 \leq i \leq n$. For any two n-sided ideals $\mathcal{L}'$ and $\mathcal{L}''$ of a Leibniz n-algebra $\mathcal{L}$, we denote by $[\mathcal{L}', \mathcal{L}'', \mathcal{L}^{n-2}]$ *the commutator ideal* of $\mathcal{L}$, that is, the n-sided ideal of $\mathcal{L}$ spanned by the brackets $[x_1, \ldots, x_n]$, where necessarily $x_i \in \mathcal{L}'$ and $x_j \in \mathcal{L}''$ for some i and j, $0 \leq i, j \leq n$, $i \neq j$. Clearly $[\mathcal{L}', \mathcal{L}'', \mathcal{L}^{n-2}] \subseteq \mathcal{L}' \cap \mathcal{L}''$.

The *centre* of a Leibniz n-algebra $\mathcal{L}$ is the n-sided ideal

$$\mathcal{Z}(\mathcal{L}) = \{x \in \mathcal{L} \mid \ [x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n] = 0 \text{ for all } x_j \in \mathcal{L}, j = 1, \ldots, \hat{i}, \ldots, n\} \, .$$

An *abelian Leibniz n-algebra* is a Leibniz n-algebra with the trivial n-bracket. It is clear that a Leibniz n-algebra $\mathcal{L}$ is abelian if and only if $\mathcal{L} = \mathcal{Z}(\mathcal{L})$.

2.2 Cohomology of Leibniz algebras

In this section we recall from [9] the main facts on the cohomological investigation of Leibniz n-algebras.

Let $\mathcal{L}$ be a Leibniz n-algebra. A *representation* of $\mathcal{L}$ (or $\mathcal{L}$-representation) is a vector space $\mathcal{M}$ together with n linear maps (n actions of $\mathcal{L}$ on $\mathcal{M}$)

$$[-, \ldots, -] : \mathcal{L}^{\otimes i} \otimes \mathcal{M} \otimes \mathcal{L}^{\otimes n-1-i} \to \mathcal{M} \ , \quad 0 \leq i \leq n-1$$

satisfying $2n-1$ equations, which are obtained from (2.1) by letting exactly one of the variables $x_1, \ldots, x_n, y_1, \ldots, y_{n-1}$ be in $\mathcal{M}$ and all the others in $\mathcal{L}$. In particular, forgetting the Leibniz n-algebra structure of $\mathcal{L}$, it can be considered as a representation of $\mathcal{L}$.

Of course, this notion for $n = 2$ coincides with the definition of a representation of a Leibniz algebra, considered in [17], where the cohomology $\mathsf{HL}^*(L, M)$ of a Leibniz algebra L with coefficients in its representation M is computed to be the cohomology of the Leibniz cochain complex $\mathsf{CL}^*(L, M)$ given by

$$\mathsf{CL}^m(L, M) = \mathrm{Hom}(L^{\otimes m}, M) \ , \quad m \geq 0$$

with the coboundary operator $\partial^m : \mathsf{CL}^m(L, M) \to \mathsf{CL}^{m+1}(L, M)$ defined by

$$\begin{aligned}(\partial^m f)(x_1, \ldots, x_{m+1}) = \ & [x_1, f(x_2, \ldots, x_{m+1})] + \sum_{2 \leq i \leq m+1} (-1)^i [f(x_1, \ldots, \widehat{x_i}, \ldots, x_{m+1}), x_i] \\ & + \sum_{1 \leq i < j \leq m+1} (-1)^{j+1} f(x_1, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, \widehat{x_j}, \ldots, x_{m+1}) \ .\end{aligned}$$

The Quillen cohomology ${}_n\mathsf{HL}^*(\mathcal{L}, \mathcal{M})$ of a Leibniz n-algebra $\mathcal{L}$ with coefficients in a representation $\mathcal{M}$ of $\mathcal{L}$ is computed in [9] as the cohomology of an explicit cochain complex ${}_n\mathsf{CL}^*(\mathcal{L}, \mathcal{M})$. An essential fact for the construction of the complex ${}_n\mathsf{CL}^*(\mathcal{L}, \mathcal{M})$ is that if $\mathcal{M}$ is a representation of a

Leibniz n-algebra $\mathcal{L}$, then $\mathrm{Hom}(\mathcal{L},\mathcal{M})$ can be considered as a representation of the Leibniz algebra $\mathcal{D}_n(\mathcal{L})$ using the following linear maps

$$[-,-] : \mathrm{Hom}(\mathcal{L},\mathcal{M}) \otimes \mathcal{D}_n(\mathcal{L}) \to \mathrm{Hom}(\mathcal{L},\mathcal{M}) \,,$$
$$[-,-] : \mathcal{D}_n(\mathcal{L}) \otimes \mathrm{Hom}(\mathcal{L},\mathcal{M}) \to \mathrm{Hom}(\mathcal{L},\mathcal{M})$$

given by

$$[f, x_1 \otimes \cdots \otimes x_{n-1}](x) = [f(x), x_1, \dots, x_{n-1}] - f[x, x_1, \dots, x_n] \,,$$
$$[x_1 \otimes \cdots \otimes x_{n-1}, f](x) = f[x, x_1, \dots, x_n] - [f(x), x_1, \dots, x_{n-1}] - \cdots - [x, x_1, \dots, f(x_{n-1})] \,.$$

Then the complex ${}_n\mathsf{CL}^*(\mathcal{L},\mathcal{M})$ is defined to be $\mathsf{CL}^*(\mathcal{D}_n(\mathcal{L}), \mathrm{Hom}(\mathcal{L},\mathcal{M}))$. Thus

$${}_n\mathsf{HL}^*(\mathcal{L},\mathcal{M}) = H^*({}_n\mathsf{CL}^*(\mathcal{L},\mathcal{M})) = \mathsf{HL}^*(\mathcal{D}_n(\mathcal{L}), \mathrm{Hom}(\mathcal{L},\mathcal{M}))\,.$$

Note that for $n = 2$ we have ${}_2\mathsf{CL}^m(\mathcal{L},\mathcal{M}) \cong \mathsf{CL}^{m+1}(\mathcal{L},\mathcal{M})$ for all $m \geq 0$ and hence

$${}_2\mathsf{HL}^m(\mathcal{L},\mathcal{M}) \cong \mathsf{HL}^{m+1}(\mathcal{L},\mathcal{M})\,.$$

Now we specify the cochains and coboundary maps of ${}_n\,\mathsf{CL}^*(\mathcal{L},\mathcal{M})$ in low dimensions, which will be useful later.
- ${}_n\mathsf{CL}^0(\mathcal{L},\mathcal{M})$ consists of all linear maps $g : \mathcal{L} \to \mathcal{M}$,
- ${}_n\mathsf{CL}^1(\mathcal{L},\mathcal{M})$ consists of all linear maps $f : \mathcal{L}^{\otimes n} \to \mathcal{M}$,
- ${}_n\mathsf{CL}^2(\mathcal{L},\mathcal{M})$ consists of all linear maps $\mathcal{L}^{\otimes 2n-1} \to \mathcal{M}$,
with the coboundary maps

$$(\partial^0 g)(x_1, \dots, x_n) = g[x_1, \dots, x_n] - \sum_{1 \leq i \leq n} [x_1, \dots, g(x_i), \dots, x_n] \,, \tag{2.2}$$

$$\begin{aligned} &(\partial^1 f)(x_1, \dots, x_n, y_1, \dots, y_{n-1}) \\ &\quad = f([x_1, \dots, x_n], y_1, \dots, y_{n-1}) - \sum_{1 \leq i \leq n} f(x_1, \dots, [x_i, y_1, \dots, y_{n-1}], \dots, x_n) \\ &\qquad + \ [f(x_1, \dots, x_n), y_1, \dots, y_{n-1}] - \sum_{1 \leq i \leq n} [x_1, \dots, f(x_i, y_1, \dots, y_{n-1}), \dots, x_n] \,, \end{aligned} \tag{2.3}$$

2.3 ${}_n\mathsf{HL}^0$ and ${}_n\mathsf{HL}^1$

Given a Leibniz n-algebra $\mathcal{L}$ and its representation $\mathcal{M}$, ${}_n\mathsf{HL}^0(\mathcal{L},\mathcal{M})$ and ${}_n\mathsf{HL}^1(\mathcal{L},\mathcal{M})$ are described in [9] via derivations and abelian extensions of Leibniz n-algebras, respectively. For future reference, we should briefly recall the relevant concepts related to these results.

A derivation from a Leibniz n-algebra $\mathcal{L}$ into a representation $\mathcal{M}$ of $\mathcal{L}$ is a linear map $d : \mathcal{L} \to \mathcal{M}$ such that

$$d[x_1, \dots, x_n] = \sum_{1 \leq i \leq n} [x_1, \dots, d(x_i), \dots, x_n],$$

for all $x_1, \ldots, x_n \in \mathcal{L}$. Let $\mathrm{Der}(\mathcal{L}, \mathcal{M})$ denote the vector space of all derivations from $\mathcal{L}$ to $\mathcal{M}$. In the case when $\mathcal{M} = \mathcal{L}$, we denote $\mathrm{Der}(\mathcal{L}, \mathcal{L})$ by $\mathrm{Der}(\mathcal{L})$. We have the following isomorphism of vector spaces [9]

$$ {}_n\mathsf{HL}^0(\mathcal{L}, \mathcal{M}) \cong \mathrm{Der}(\mathcal{L}, \mathcal{M}). $$

Let

$$ 0 \longrightarrow \mathcal{M} \xrightarrow{\sigma} \mathcal{E} \xrightarrow{\pi} \mathcal{L} \longrightarrow 0 $$

be a short exact sequence of Leibniz n-algebras, where $\mathcal{M}$ is a representation of $\mathcal{L}$ considered as an abelian Leibniz n-algebra. Take a linear section s of π, i. e. a linear map $s : \mathcal{L} \to \mathcal{E}$ such that $\pi s = \mathrm{id}_{\mathcal{L}}$. Then there is an induced $\mathcal{L}$-representation structure on $\mathcal{M}$ defined in the standard way by taking the n-ary bracket in $\mathcal{E}$, that is

$$ [x_1, \ldots, x_{i-1}, m, x_{i+1}, \ldots, x_n] = \sigma^{-1}\big[s(x_1), \ldots, s(x_{i-1}), \sigma(m), s(x_{i+1}), \ldots, s(x_n)\big], \tag{2.4} $$

for $x_1, \cdots, x_n \in \mathcal{L}$ and $i \in \{1, \cdots, n\}$. Since $\mathcal{M}$ is abelian, it is clear that the definition does not depend on the linear section s.

An abelian extension of a Leibniz n-algebra $\mathcal{L}$ by its representation $\mathcal{M}$ is a short exact sequence of Leibniz n-algebras

$$ \mathsf{E} : 0 \longrightarrow \mathcal{M} \xrightarrow{\sigma} \mathcal{E} \xrightarrow{\pi} \mathcal{L} \longrightarrow 0, $$

such that $[e_1, \ldots, e_n] = 0$ in $\mathcal{E}$ as soon as $e_i, e_j \in \mathcal{M}$, $1 \leq i \neq j \leq n$ and the induced $\mathcal{L}$-representation structure on $\mathcal{M}$ coincides with the given one. Two such abelian extensions E and E' of $\mathcal{L}$ by $\mathcal{M}$ are isomorphic if there is a homomorphism of Leibniz n-algebras $\mathcal{E} \to \mathcal{E}'$ which, together with the identity maps on $\mathcal{M}$ and $\mathcal{L}$, forms commutative squares.

The set $\mathrm{Ext}(\mathcal{L}, \mathcal{M})$ is not empty since it contains the class of the abelian extension defined by the semi-direct product of $\mathcal{M}$ and $\mathcal{L}$, i. e.

$$ \mathsf{E}_0 : 0 \longrightarrow \mathcal{M} \xrightarrow{\sigma_0} \mathcal{M} \rtimes \mathcal{L} \xrightarrow{\pi_0} \mathcal{L} \longrightarrow 0. \tag{2.5} $$

Here we recall from [5] that *the semi-direct product* $\mathcal{M} \rtimes \mathcal{L}$ is the Leibniz n-algebra with the underlying vector space $\mathcal{M} \oplus \mathcal{L}$ and n-ary bracket given by

$$ [(m_1, x_1), \ldots, (m_n, x_n)] = \big(\sum_{1 \leq i \leq n} [x_1, \ldots, x_{i-1}, m_i, x_{i+1}, \ldots, x_n], [x_1, \ldots, x_n] \big). $$

σ_0 and π_0 in the sequence (2.5) are defined by $\sigma_0(m) = (m, 0)$ and $\pi_0(m, x) = x$. Moreover, this sequence splits by the homomorphism of Leibniz n-algebras $\mathcal{L} \to \mathcal{M} \rtimes \mathcal{L}$, $x \mapsto (0, x)$.

Any abelian extension $\mathsf{E} : 0 \longrightarrow \mathcal{M} \xrightarrow{\sigma} \mathcal{E} \xrightarrow{\pi} \mathcal{L} \longrightarrow 0$ gives rise to a 1-cocycle $f \in {}_n\mathsf{CL}^1(\mathcal{L}, \mathcal{M})$ by choosing a linear section $s : \mathcal{L} \to \mathcal{E}$ of π, and by defining $f : \mathcal{L}^{\otimes n} \to \mathcal{M}$ as follows

$$ f(x_1, \cdots, x_n) = \sigma^{-1}\big([s(x_1), \cdots, s(x_n)] - s[x_1, \cdots, x_n]\big), \tag{2.6} $$

for all $x_1, \cdots, x_n \in \mathcal{L}$. This gives a well-defined bijection between the set of equivalence classes $\mathrm{Ext}(\mathcal{L}, \mathcal{M})$ of such abelian extensions of $\mathcal{L}$ by $\mathcal{M}$ and the first cohomology of $\mathcal{L}$ with coefficients in $\mathcal{M}$ [9], i.e.

$$ \mathrm{Ext}(\mathcal{L}, \mathcal{M}) \cong {}_n\mathsf{HL}^1(\mathcal{L}, \mathcal{M}). $$

Let us note that this bijection maps the class of the trivial extension E_0 to the zero element in ${}_n\mathsf{HL}^1(\mathcal{L},\mathcal{M})$. Moreover, it allows us to endow the set $\operatorname{Ext}(\mathcal{L},\mathcal{M})$ with a vector space structure induced from the one of ${}_n\mathsf{HL}^1(\mathcal{L},\mathcal{M})$. The respective addition in $\operatorname{Ext}(\mathcal{L},\mathcal{M})$ is defined by the well-known "Baer sum" of extensions and the scalar multiplication is defined for any $\lambda \in \mathbb{K}$ by $\lambda\,\mathrm{cl}(\mathsf{E}) = \mathrm{cl}\,(\lambda\,\mathsf{E})$, where $\lambda\,\mathsf{E} : 0 \to \mathcal{M} \xrightarrow{\lambda\sigma} \mathcal{E} \xrightarrow{\pi} \mathcal{L} \to 0$.

3 Derivations of abelian extensions of Leibniz n-algebras

3.1 Extensibility of derivations

In this section, we formulate and solve the extensibility problem of derivations associated with an abelian extension of Leibniz n-algebras.

Definition 3.1. Let $0 \longrightarrow \mathcal{M} \xrightarrow{\sigma} \mathcal{E} \xrightarrow{\pi} \mathcal{L} \longrightarrow 0$ be an abelian extension of a Leibniz n-algebra $\mathcal{L}$ by its representation $\mathcal{M}$. A pair $(d_{\mathcal{M}}, d_{\mathcal{L}}) \in \operatorname{Der}(\mathcal{M}) \times \operatorname{Der}(\mathcal{L})$ is called extensible if there is a derivation $d_{\mathcal{E}} \in \operatorname{Der}(\mathcal{E})$ such that $d_{\mathcal{E}}\sigma = \sigma d_{\mathcal{M}}$ and $\pi d_{\mathcal{E}} = d_{\mathcal{L}}\pi$.

Let us remark that, here $\mathcal{M}$ is considered as an abelian Leibniz n-algebra and hence, a derivation $d_{\mathcal{M}} \in \operatorname{Der}(\mathcal{M})$ is just a linear map, and $\operatorname{Der}(\mathcal{M}) = \operatorname{End}_{\mathbb{K}}(\mathcal{M})$.

Lemma 3.2. Let $\mathcal{M}$ be a representation of a Leibniz n-algebra $\mathcal{L}$. Consider $\mathcal{M}$ as an abelian Leibniz n-algebra. Let $d_{\mathcal{L}} \in \operatorname{Der}(\mathcal{L})$ and $d_{\mathcal{M}} \in \operatorname{Der}(\mathcal{M})$. Then $(d_{\mathcal{M}}, d_{\mathcal{L}}) \in \operatorname{Der}(\mathcal{M} \rtimes \mathcal{L})$ if and only if the following equality holds for all $i \in \{1, \cdots, n\}$, $x_j \in \mathcal{L}$ and $m \in \mathcal{M}$

$$d_{\mathcal{M}}[x_1, \dots, x_{i-1}, m, x_{i+1}, \dots, x_n] = \sum_{1 \le j \ne i \le n} [x_1, \dots, d_{\mathcal{L}}(x_j), \dots, x_{i-1}, m, x_{i+1}, \dots, x_n] + [x_1, \dots, x_{i-1}, d_{\mathcal{M}}(m), x_{i+1}, \dots, x_n] \tag{3.1}$$

Proof. The proof requires only direct calculations. Q.E.D.

For any Leibniz n-algebra $\mathcal{L}$ and its representation $\mathcal{M}$, let us denote by

$$\mathrm{D}(\mathcal{M},\mathcal{L}) = \big\{(d_{\mathcal{M}}, d_{\mathcal{L}}) \in \operatorname{Der}(\mathcal{M}) \times \operatorname{Der}(\mathcal{L}) \mid d_{\mathcal{M}} \text{ and } d_{\mathcal{L}} \text{ satisfy equation (3.1)}\big\}.$$

Let us remark that $\mathrm{D}(\mathcal{M},\mathcal{L})$ is a vector subspace of $\operatorname{Der}(\mathcal{M} \rtimes \mathcal{L})$ by Lemma 3.2.

Now we define a map

$$\theta : \mathrm{D}(\mathcal{M},\mathcal{L}) \times {}_n\mathsf{CL}^1(\mathcal{L},\mathcal{M}) \longrightarrow {}_n\mathsf{CL}^1(\mathcal{L},\mathcal{M}) \tag{3.2}$$

as follows. For any $(d_{\mathcal{M}}, d_{\mathcal{L}}) \in \mathrm{D}(\mathcal{M},\mathcal{L})$ and any 1-cochain $f \in {}_n\mathsf{CL}^1(\mathcal{L},\mathcal{M})$ we set

$$\theta\big((d_{\mathcal{M}}, d_{\mathcal{L}}), f\big) = f_\theta,$$

where $f_\theta : \mathcal{L}^{\otimes n} \to \mathcal{M}$ is given by

$$f_\theta(x_1, \dots, x_n) = d_{\mathcal{M}} f(x_1, \dots, x_n) - \sum_{1 \le i \le n} f(x_1, \dots, d_{\mathcal{L}}(x_i), \dots, x_n). \tag{3.3}$$

Lemma 3.3. Let us fix $(d_{\mathcal{M}}, d_{\mathcal{L}}) \in \mathrm{D}(\mathcal{M}, \mathcal{L})$, then we have:

(i) If $f \in {}_n\mathsf{CL}^1(\mathcal{L}, \mathcal{M})$ is a 1-cocycle, then f_θ is a 1-cocycle as well.

(ii) If f and f' are two cohomologous cocycles, then f_θ and f'_θ are cohomologous as well.

Proof. (i) We need to check that $(\partial^1 f_\theta)(x_1, \dots, x_n, y_1, \dots, y_{n-1}) = 0$, by (2.3) this means

$$\begin{aligned} f_\theta([x_1, \dots, x_n], y_1, \dots, y_{n-1}) + {} & [f_\theta(x_1, \dots, x_n), y_1, \dots, y_{n-1}] \\ = \sum_{1 \le i \le n} & \Big(f_\theta(x_1, \dots, [x_i, y_1, \dots, y_{n-1}], \dots, x_n) + [x_1, \dots, f_\theta(x_i, y_1, \dots, y_{n-1}), \dots, x_n] \Big) , \end{aligned}$$

whenever the same conditions hold for f. This requires routine calculations using the equations (3.1) and (3.3).

(ii) If f and f' are cohomologous cocycles, by (2.2) there is a linear map $g : \mathcal{L} \to \mathcal{M}$ such that

$$(f - f')(x_1, \dots, x_n) = g[x_1, \dots, x_n] - \sum_{1 \le i \le n} [x_1, \dots, g(x_i), \dots, x_n].$$

Then using only the definition of f_θ in (3.3), straightforward computations show that the linear map $d_{\mathcal{M}} g - g d_{\mathcal{L}} : \mathcal{L} \to \mathcal{M}$ satisfies the condition

$$f_\theta - f'_\theta = \partial^0 (d_{\mathcal{M}} g - g d_{\mathcal{L}}).$$

This completes the proof. Q.E.D.

Remark 3.4. As a consequence of Lemma 3.3, note that the map θ in (3.2) induces a bilinear map

$$\Theta : \mathrm{D}(\mathcal{M}, \mathcal{L}) \times {}_n\mathsf{HL}^1(\mathcal{L}, \mathcal{M}) \longrightarrow {}_n\mathsf{HL}^1(\mathcal{L}, \mathcal{M}).$$

Using Lemma 3.3 we state the following definition.

Definition 3.5. Let $\mathsf{E} : 0 \longrightarrow \mathcal{M} \xrightarrow{\sigma} \mathcal{E} \xrightarrow{\pi} \mathcal{L} \longrightarrow 0$ be an abelian extension of a Leibniz n-algebra $\mathcal{L}$ by its representation $\mathcal{M}$ and $f \in {}_n\mathsf{CL}^1(\mathcal{L}, \mathcal{M})$ be the induced 1-cocycle as in (2.6). The map

$$\omega : \mathrm{D}(\mathcal{M}, \mathcal{L}) \longrightarrow {}_n\mathsf{HL}^1(\mathcal{L}, \mathcal{M}), \quad \omega(d_{\mathcal{M}}, d_{\mathcal{L}}) = \Theta\big((d_{\mathcal{M}}, d_{\mathcal{L}}), \mathrm{cl}(f)\big) = \mathrm{cl}(f_\theta)$$

is called the Wells map associated to the given abelian extension E.

Remark 3.6. Note that if the abelian extension E is split, that is, there is a homomorphism of Leibniz n-algebras $s : \mathcal{L} \to \mathcal{E}$ such that $\pi s = \mathrm{id}_{\mathcal{L}}$, then $f_\theta = f = 0$ and so ω is the trivial map.

Theorem 3.7. Let $\mathsf{E} : 0 \longrightarrow \mathcal{M} \xrightarrow{\sigma} \mathcal{E} \xrightarrow{\pi} \mathcal{L} \longrightarrow 0$ be an abelian extension of a Leibniz n-algebra $\mathcal{L}$ by its representation $\mathcal{M}$. A pair $(d_{\mathcal{M}}, d_{\mathcal{L}}) \in \mathrm{Der}(\mathcal{M}) \times \mathrm{Der}(\mathcal{L})$ is extensible if and only if $(d_{\mathcal{M}}, d_{\mathcal{L}}) \in \mathrm{D}(\mathcal{M}, \mathcal{L})$ and $w(d_{\mathcal{M}}, d_{\mathcal{L}}) = 0$.

Proof. Let $s : \mathcal{L} \to \mathcal{E}$ be a linear section of π. Then any element of $\mathcal{E}$ has the form $\sigma(m) + s(x)$, where $m \in \mathcal{M}$ and $x \in \mathcal{L}$. To simplify notations, we write m instead of $\sigma(m) \in \mathcal{E}$.

First we show that $(d_{\mathcal{M}}, d_{\mathcal{L}}) \in \mathrm{Der}(\mathcal{M}) \times \mathrm{Der}(\mathcal{L})$ is extensible if $(d_{\mathcal{M}}, d_{\mathcal{L}}) \in \mathrm{D}(\mathcal{M}, \mathcal{L})$ and $w(d_{\mathcal{M}}, d_{\mathcal{L}}) = \theta\big((d_{\mathcal{L}}, d_{\mathcal{M}}), \mathrm{cl}(f)\big) = \mathrm{cl}(f_\theta) = 0$. Thus, there exists a linear map $g : \mathcal{L} \to \mathcal{M}$ such that $f_\theta = \partial^0(g)$. By using (2.2) and (3.3) we get immediately:

$$\begin{aligned} d_{\mathcal{M}} f(x_1, \ldots, x_n) - &\sum_{1\le i\le n} f(x_1, \ldots, d_{\mathcal{L}}(x_i), \ldots, x_n) \\ &= g[x_1, \ldots, x_n] - \sum_{1\le i\le n} [x_1, \ldots, g(x_i), \ldots, x_n]. \end{aligned} \tag{3.4}$$

Let us define $d_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$ by

$$d_{\mathcal{E}}(m + s(x)) = d_{\mathcal{M}}(m) - g(x) + s d_{\mathcal{L}}(x). \tag{3.5}$$

Obviously $d_{\mathcal{E}}$ is a linear map and

$$\begin{aligned} &d_{\mathcal{E}}(m) = d_{\mathcal{M}}(m), \\ &\pi d_{\mathcal{E}}(m + s(x)) = \pi d_{\mathcal{M}}(m) - \pi g(x) + \pi s d_{\mathcal{L}}(x) = d_{\mathcal{L}}(x) = d_{\mathcal{L}} \pi(m + s(x)). \end{aligned}$$

To see that $d_{\mathcal{E}}$ is a derivation, it suffices to check the following identities

$$\begin{aligned} d_{\mathcal{E}}&\big[s(x_1), \ldots, s(x_{i-1}), m, s(x_{i+1}), \ldots, s(x_n)\big] \\ &= \sum_{1\le j\ne i\le n} [s(x_1), \ldots, d_{\mathcal{E}} s(x_j), \ldots, s(x_{i-1}), m, s(x_{i+1}), \ldots, s(x_n)] \\ &\qquad + \big[s(x_1), \ldots, s(x_{i-1}), d_{\mathcal{E}}(m), s(x_{i+1}), \ldots, s(x_n)\big] \end{aligned} \tag{3.6}$$

and

$$d_{\mathcal{E}}\big[s(x_1), \ldots, s(x_n)\big] = \sum_{1\le i\le n} \big[s(x_1), \ldots, d_{\mathcal{E}} s(x_i), \ldots, s(x_n)\big]. \tag{3.7}$$

Using the fact that $d_{\mathcal{E}}(m) = d_{\mathcal{M}}(m)$ and $d_{\mathcal{E}}(x_j) = -g(x_j) + s d_{\mathcal{L}}(x_j)$, the identity (3.6) easily follows from (3.1). To prove (3.7), we present the following calculations:

$$
\begin{aligned}
d_{\mathcal{E}}[s(x_1),\dots,s(x_n)] &\overset{(2.6)}{=} d_{\mathcal{E}}\big(f(x_1,\dots,x_n)+s[x_1,\dots,x_n]\big)\\
&\overset{(3.5)}{=} d_{\mathcal{M}}f(x_1,\dots,x_n)-g[x_1,\dots,x_n]+sd_{\mathcal{L}}[x_1,\dots,x_n]\\
&=d_{\mathcal{M}}f(x_1,\dots,x_n)-g[x_1,\dots,x_n]+\sum_{1\le i\le n}s\big[x_1,\dots,d_{\mathcal{L}}(x_i),\dots,x_n)\big]\\
&\overset{(2.6)}{=} d_{\mathcal{M}}f(x_1,\dots,x_n)-g[x_1,\dots,x_n]-\sum_{1\le i\le n}f(x_1,\dots,d_{\mathcal{L}}(x_i),\dots,x_n)\\
&\qquad\qquad +\sum_{1\le i\le n}\big[s(x_1),\dots,sd_{\mathcal{L}}(x_1),\dots,s(x_i)\big]\\
&\overset{(3.4)}{=} -\sum_{1\le i\le n}\big[x_1,\dots,g(x_i),\dots,x_n\big]+\sum_{1\le i\le n}\big[s(x_1),\dots,sd_{\mathcal{L}}(x_i),\dots,s(x_i)\big]\\
&\overset{(2.4)}{=} -\sum_{1\le i\le n}\big[s(x_1),\dots,g(x_i),\dots,s(x_n)\big]+\sum_{1\le i\le n}\big[s(x_1),\dots,sd_{\mathcal{L}}(x_i),\dots,s(x_i)\big]\\
&\overset{(3.5)}{=} \sum_{1\le i\le n}\big[s(x_1),\dots,d_{\mathcal{E}}s(x_i),\dots,s(x_n)\big].
\end{aligned}
$$

Conversely, suppose that $(d_{\mathcal{M}},d_{\mathcal{L}})\in \mathrm{Der}(\mathcal{M})\times\mathrm{Der}(\mathcal{L})$ is extensible, i. e. there exists $d_{\mathcal{E}}\in \mathrm{Der}(\mathcal{E})$ such that $d_{\mathcal{E}}\sigma=\sigma d_{\mathcal{M}}$ and $\pi d_{\mathcal{E}}=d_{\mathcal{L}}\pi$, then $d_{\mathcal{M}}$ is the restriction of $d_{\mathcal{E}}$ on $\mathcal{M}$, $d_{\mathcal{M}}=d_{\mathcal{E}}|_{\mathcal{M}}$. Fix again a linear section $s:\mathcal{L}\to\mathcal{E}$ of π. Since $sd_{\mathcal{L}}(x)-d_{\mathcal{E}}s(x)\in\operatorname{Ker}\pi$ for all $x\in\mathcal{L}$, we have a linear map $g:\mathcal{L}\to\mathcal{M}$ defined by

$$
g(x)=sd_{\mathcal{L}}(x)-d_{\mathcal{E}}s(x). \tag{3.8}
$$

Now we claim that the pair $(d_{\mathcal{M}},d_{\mathcal{L}})\in \mathrm{D}(\mathcal{M},\mathcal{L})$, i. e. $d_{\mathcal{M}}$ and $d_{\mathcal{L}}$ satisfy the condition (3.1). In effect,

$$
\begin{aligned}
d_{\mathcal{M}}\big[x_1,\dots,x_{i-1},m,x_{i+1},\dots,x_n\big] &= d_{\mathcal{E}}\big[s(x_1),\dots,s(x_{i-1}),m,s(x_{i+1}),\dots,s(x_n)\big]\\
&=\sum_{1\le j\ne i\le n}[s(x_1),\dots,d_{\mathcal{E}}s(x_j),\dots,s(x_{i-1}),m,s(x_{i+1}),\dots,s(x_n)]\\
&\qquad +[s(x_1),\dots,s(x_{i-1}),d_{\mathcal{E}}(m),s(x_{i+1}),\dots,s(x_n)]\\
&\overset{(3.8)}{=}\sum_{1\le j\ne i\le n}[s(x_1),\dots,sd_{\mathcal{L}}(x_j)-g(x_j),\dots,s(x_{i-1}),m,s(x_{i+1}),\dots,s(x_n)]\\
&\qquad +[x_1,\dots,x_{i-1},d_{\mathcal{M}}(m),x_{i+1},\dots,x_n]\\
&=\sum_{1\le j\ne i\le n}[s(x_1),\dots,sd_{\mathcal{L}}(x_j),\dots,s(x_{i-1}),m,s(x_{i+1}),\dots,s(x_n)]\\
&\qquad +[x_1,\dots,x_{i-1},d_{\mathcal{M}}(m),x_{i+1},\dots,x_n]\\
&=\sum_{1\le j\ne i\le n}[x_1,\dots,d_{\mathcal{L}}(x_j),\dots,x_{i-1},m,x_{i+1},\dots,x_n]+[x_1,\dots,x_{i-1},d_{\mathcal{M}}(m),x_{i+1},\dots,x_n].
\end{aligned}
$$

It remains to show that $w(d_{\mathcal{M}}, d_{\mathcal{L}}) = \theta\big((d_{\mathcal{L}}, d_{\mathcal{M}}), \mathrm{cl}(f)\big) = \mathrm{cl}(f_\theta) = 0$. For this, let's check that $f_\theta = \partial^0(g)$, where g is defined in (3.8). In effect, for any $x_1, \dots, x_n \in \mathcal{L}$ we have

$$\begin{aligned} d_{\mathcal{E}}\big[s(x_1), \dots, s(x_n)\big] &\overset{(2.6)}{=} d_{\mathcal{E}}\big(s[x_1 \dots, x_n] + f(x_1 \dots, x_n)\big) \\ &\overset{(3.8)}{=} sd_{\mathcal{L}}[x_1 \dots, x_n] - g[x_1 \dots, x_n] + d_{\mathcal{M}} f(x_1 \dots, x_n). \end{aligned} \tag{3.9}$$

On the other hand, since $d_{\mathcal{E}}$ is a derivation, we get

$$\begin{aligned} d_{\mathcal{E}}\big[s(x_1), \dots, s(x_n)\big] &= \sum_{1\le i\le n} \big[s(x_1), \dots, d_{\mathcal{E}} s(x_i), \dots, s(x_n)\big] \\ &\overset{(3.8)}{=} \sum_{1\le i\le n} \big[s(x_1), \dots, sd_{\mathcal{L}}(x_i), \dots, s(x_n)\big] - \sum_{1\le i\le n} \big[s(x_1), \dots, g(x_i), \dots, s(x_n)\big] \\ &\overset{(2.6)}{=} s\Big(\sum_{1\le i\le n} \big[x_1, \dots, d_{\mathcal{L}}(x_i), \dots, x_n\big]\Big) + \sum_{1\le i\le n} f(x_1, \dots, d_{\mathcal{L}}(x_i), \dots, x_n) \\ &\qquad - \sum_{1\le i\le n} \big[s(x_1), \dots, g(x_i), \dots, s(x_n)\big] \\ &= sd_{\mathcal{L}}[x_1 \dots, x_n] + \sum_{1\le i\le n} f(x_1, \dots, d_{\mathcal{L}}(x_i), \dots, x_n) - \sum_{1\le i\le n} \big[s(x_1), \dots, g(x_i), \dots, s(x_n)\big]. \end{aligned} \tag{3.10}$$

Comparing the last lines in (3.9) and (3.10), we easily get

$$\begin{aligned} f_\theta(x_1, \dots, x_n) &= d_{\mathcal{M}} f(x_1, \dots, x_n) - \sum_{1\le i\le n} f(x_1, \dots, d_{\mathcal{L}}(x_i), x_n) \\ &\overset{(3.9),(3.10)}{=} g[x_1 \dots, x_n] - \sum_{1\le i\le n} \big[s(x_1), \dots, g(x_i), s(x_n)\big] \\ &\overset{(2.4)}{=} g[x_1 \dots, x_n] - \sum_{1\le i\le n} \big[x_1, \dots, g(x_i), x_n\big] \overset{(2.2)}{=} \partial^0(g)(x_1 \dots, x_n). \end{aligned}$$

This completes the proof. Q.E.D.

3.2 The Wells sequence for derivations of Leibniz n-algebras

Given an abelian extension $\mathsf{E} : 0 \longrightarrow \mathcal{M} \xrightarrow{\sigma} \mathcal{E} \xrightarrow{\pi} \mathcal{L} \longrightarrow 0$ of a Leibniz n-algebra $\mathcal{L}$ by its representation $\mathcal{M}$, we fix a linear section s of π and introduce the following notation

$$\mathrm{Der}(\mathcal{E} \,|\, \mathcal{M}) = \{d_{\mathcal{E}} \in \mathrm{Der}(\mathcal{E}) \mid d_{\mathcal{E}}(\mathcal{M}) \subseteq \mathcal{M}\}.$$

One readily checks that any element $d_{\mathcal{E}} \in \mathrm{Der}(\mathcal{E} \,|\, \mathcal{M})$ defines two derivations of Leibniz n-algebras

$$d_{\mathcal{E}|\mathcal{M}} = d_{\mathcal{E}}|_{\mathcal{M}} \in \mathrm{Der}(\mathcal{M}) \quad \text{and} \quad d^{\mathcal{E}}_{\mathcal{L}} = \pi d_{\mathcal{E}} s \in \mathrm{Der}(\mathcal{L}).$$

Moreover, since $\pi d_{\mathcal{E}}(m) = 0$ for any $m \in \mathcal{M}$, it follows that $d^{\mathcal{E}}_{\mathcal{L}}$ does not depend on the choice of the section s. Thus, we get a linear map

$$\kappa : \mathrm{Der}(\mathcal{E} \,|\, \mathcal{M}) \longrightarrow \mathrm{Der}(\mathcal{M}) \times \mathrm{Der}(\mathcal{L}), \quad \kappa(d_{\mathcal{E}}) = (d_{\mathcal{E}|\mathcal{M}}, d^{\mathcal{E}}_{\mathcal{L}}).$$

It is easy to see that $\sigma d_{\mathcal{E}|\mathcal{M}} = d_{\mathcal{E}}\sigma$ and $d^{\mathcal{E}}_{\mathcal{L}}\pi = \pi d_{\mathcal{E}}$, that is, $(d_{\mathcal{E}|\mathcal{M}}, d^{\mathcal{E}}_{\mathcal{L}})$ is an extensible pair in $\mathrm{Der}(\mathcal{M}) \times \mathrm{Der}(\mathcal{L})$. Then by Theorem 3.7 we have that $(d_{\mathcal{E}|\mathcal{M}}, d^{\mathcal{E}}_{\mathcal{L}}) \in \mathrm{D}(\mathcal{M}, \mathcal{L})$ and $w(d_{\mathcal{E}|\mathcal{M}}, d^{\mathcal{E}}_{\mathcal{L}}) = 0$. Moreover, we have the following result

Lemma 3.8. With the above notations, the following assertions hold:

(i) $\mathrm{Im}(\kappa) \subseteq \mathrm{D}(\mathcal{M}, \mathcal{L})$;

(ii) $\mathrm{Im}(\kappa) = \mathrm{Ker}(\omega)$.

Proof. (i) There is nothing to prove because as we have shown above $\kappa(d_{\mathcal{E}}) \in \mathrm{D}(\mathcal{M}, \mathcal{L})$ for all $d_{\mathcal{E}} \in \mathrm{Der}(\mathcal{E} \,|\, \mathcal{M})$.

(ii) Since $w\kappa(d_{\mathcal{E}}) = w(d_{\mathcal{E}|\mathcal{M}}, d^{\mathcal{E}}_{\mathcal{L}}) = 0$ for all $d_{\mathcal{E}} \in \mathrm{Der}(\mathcal{M}, \mathcal{L})$, we have $\mathrm{Im}(\kappa) \subseteq \mathrm{Ker}(\omega)$. So we just need to show that $\mathrm{Ker}(\omega) \subseteq \mathrm{Im}(\kappa)$. For that, take $(d_{\mathcal{M}}, d_{\mathcal{L}}) \in \mathrm{Ker}(\omega)$. Then $\theta\big((d_{\mathcal{M}}, d_{\mathcal{L}}), \mathrm{cl}(f)\big) = 0$, where f is the 1-cocycle in ${}_n\mathsf{CL}^*(\mathcal{L}, \mathcal{M})$ induced by the given extension E as in (2.6). So, there exists a linear map $\big(0\text{-cochain in } {}_n\mathsf{CL}^*(\mathcal{L}, \mathcal{M})\big)$ $g : \mathcal{L} \to \mathcal{M}$ such that $\theta\big((d_{\mathcal{M}}, d_{\mathcal{L}}), \mathrm{cl}(f)\big) = \mathrm{cl}(\partial^0(g))$, i. e. (3.4) holds. We define $d_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$ in the same way as in (3.5), $d_{\mathcal{E}}(m + s(x)) = d_{\mathcal{M}}(m) - g(x) + s d_{\mathcal{L}}(x)$. One can repeat the respective part of the proof of Theorem 3.7 to show that $d_{\mathcal{E}} \in \mathrm{Der}(\mathcal{E})$. Moreover, it is obvious that $d_{\mathcal{E}}(\mathcal{M}) \subseteq \mathcal{M}$ and hence, $d_{\mathcal{E}} \in \mathrm{Der}(\mathcal{E} \,|\, \mathcal{M})$. At the same time, $d_{\mathcal{E}}|_{\mathcal{M}} = d_{\mathcal{M}}$ and $\pi d_{\mathcal{E}} s = d_{\mathcal{L}}$, which means that $\kappa(d_{\mathcal{E}}) = (d_{\mathcal{L}}, d_{\mathcal{E}})$ and the proof is completed. Q.E.D.

Let us denote by $Z^0(\mathcal{L}, \mathcal{M})$ the vector space of all 0-cocycles in the cochain complex ${}_n\mathsf{CL}^*(\mathcal{L}, \mathcal{M})$, i. e. it consists of all linear maps $g : \mathcal{L} \to \mathcal{M}$ satisfying

$$g[x_1, \dots, x_n] = \sum_{1 \le i \le n} [x_1, \dots, g(x_i), \dots, x_n] \tag{3.11}$$

for all $x_1, \dots, x_n \in \mathcal{L}$.

Theorem 3.9. Let $\mathsf{E} : 0 \longrightarrow \mathcal{M} \xrightarrow{\sigma} \mathcal{E} \xrightarrow{\pi} \mathcal{L} \longrightarrow 0$ be an abelian extension of a Leibniz n-algebra $\mathcal{L}$ by its representation $\mathcal{M}$. Then there is an exact sequence of vector spaces

$$0 \longrightarrow Z^0(\mathcal{L}, \mathcal{M}) \xrightarrow{\tau} \mathrm{Der}(\mathcal{E} \,|\, \mathcal{M}) \xrightarrow{\kappa} \mathrm{D}(\mathcal{M}, \mathcal{L}) \xrightarrow{\omega} {}_n\mathsf{HL}^1(\mathcal{L}, \mathcal{M}). \tag{3.12}$$

Proof. Thanks to Lemma 3.8 (ii), it suffices to construct an injection of vector spaces $\tau : Z^0(\mathcal{L}, \mathcal{M}) \to \mathrm{Der}(\mathcal{E} \,|\, \mathcal{M})$ and show exactness at the term $\mathrm{Der}(\mathcal{E} \,|\, \mathcal{M})$.

We define τ as follows. For any $g \in Z^0(\mathcal{L}, \mathcal{M})$, let $\tau(g) = d_{\mathcal{E}}$, where $d_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$ is given by $d_{\mathcal{E}}(m + s(x)) = g(x)$, for any $m \in \mathcal{M}$, $x \in \mathcal{L}$. Clearly $d_E|_{\mathcal{M}} = 0$ and $d^{\mathcal{E}}_{\mathcal{L}}(x) = \pi d_{\mathcal{E}} s(x) = \pi g(x) = 0$. Moreover, using (3.11) it is easy to check that $d_{\mathcal{E}}$ is a derivation of Leibniz n-algebras. Hence $\tau(g) = d_{\mathcal{E}} \in \mathrm{Der}(\mathcal{E} \,|\, \mathcal{M})$ and $\kappa(d_{\mathcal{E}}) = (0, 0)$, i.e. $\mathrm{Im}(\tau) \subseteq \mathrm{Ker}(\kappa)$. Obviously $\mathrm{Ker}(\tau) = 0$, so it is an injection.

It remains to check that $\mathrm{Ker}(\kappa) \subseteq \mathrm{Im}(\tau)$. For that, take $d_{\mathcal{E}} \in \mathrm{Der}(\mathcal{E} \,|\, \mathcal{M})$ such that $\kappa(d_{\mathcal{E}}) = (0, 0)$, i. e. $d_{\mathcal{E}}|_{\mathcal{M}} = 0$ and $\pi d_{\mathcal{E}} s = 0$. For any $x \in \mathcal{L}$ we get $d_{\mathcal{E}} s(x) \in \mathrm{Ker}(\pi) = \mathcal{M}$. Thus there is a linear map $g : \mathcal{L} \to \mathcal{M}$ given by $g = d_{\mathcal{E}} s$. One can easily check that g does not depend on the choice of the linear section s, it satisfies condition (3.11), i. e. $g \in Z^0(\mathcal{L}, \mathcal{M})$, and $\sigma(g) = d_{\mathcal{E}}$. Q.E.D.

Finally, let us note that, if the abelian extension $0 \longrightarrow \mathcal{M} \xrightarrow{\sigma} \mathcal{E} \xrightarrow{\pi} \mathcal{L} \longrightarrow 0$ is split, i.e. there is a homomorphism of Leibniz n-algebras $s : \mathcal{L} \to \mathcal{E}$ such that $\pi s = \mathrm{id}_{\mathcal{L}}$, then $w = 0$ by Remark 3.6. Hence, the exact sequence (3.12) gives the following isomorphism of vector spaces

$$\mathrm{Der}(\mathcal{E} \,|\, \mathcal{M}) \cong \mathrm{D}(\mathcal{M}, \mathcal{L}) \times Z^0(\mathcal{L}, \mathcal{M}).$$

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