Inverse source problem for a one-dimensional degenerate hyperbolic problem

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Abstract

This article focuses on solving inverse problems related to identifying spatial components in a one-dimensional degenerate hyperbolic equation. The study is critical in fields like physics, engineering, and environmental science. We establish the well-posedness of the direct problem and provide essential results on solving the inverse problem using functional minimization and Tikhonov regularization for accuracy. Two algorithms are introduced for different scenarios: a descent algorithm for cases with reference data and a "Thresholding" algorithm for situations without reference. Experimental validation shows the effectiveness of these methods in accurately recovering unknown source terms. This research advances understanding in hyperbolic equation inverse problems and provides practical tools for real-world applications.

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1 Introduction and main results

Hyperbolic partial differential equations (PDEs) play a central role in a wide range of scientific and engineering disciplines. Their fundamental mathematical properties, which include finite solution propagation and the Huygens principle, make them indispensable tools for modeling various natural phenomena, such as wave dynamics. These equations go beyond the boundaries of their respective fields and prove valuable in both physics and engineering. An essential aspect of mathematical sciences involves deriving governing equations for systems from existing solution data, especially in ill-posed problems. This process is crucial for understanding the underlying mechanisms of specific physical phenomena [29].

In various scientific and engineering domains, including cosmology [18], data science [27], remote sensing [34], medicine [19], and geophysics [20], the solution of inverse problems is essential. These problems aim to determine unknown coefficients within partial differential equations, relying on limited temporal knowledge of the system.

Recent research has been directed towards degenerate wave equations, which possess the versatility to model a wide range of physical scenarios. These scenarios encompass image processing [39, 37, 38], biomedical imaging [11, 4, 42], non-destructive testing [12, 21, 32], radar and sonar imaging [40, 3], as well as applications in porous media, laminar flow, climate models, population genetics, and financial mathematics [7, 5].

This study builds upon the investigation introduced in [6], where the challenge of identifying initial conditions for degenerate wave problems was addressed. The authors employed two distinctive approaches: regularization by viscose-elasticity and Tikhonov's regularization.

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A notable characteristic of these challenges lies in their remarkable sensitivity to perturbations in measurements, which classifies them as ill-posed problems. To ensure the stability of results, prudent regularization methods are often indispensable. The conventional approach to addressing these issues, which involves minimizing a data fitting function, encounters obstacles such as non-convexity and the presence of numerous local minima.

In certain imaging applications, the demand for real-time results makes traditional optimization methods impractical. Consequently, a reevaluation or streamlining of the employed modeling and approaches becomes imperative [13, 14, 17, 16, 25, 28].

The core of our study focuses on the identification of the source term in a degenerate hyperbolic equation that exhibits degeneracy within the domain. To be more precise, we consider the following problem:

$$\begin{cases} \partial_{tt}u(x,t) - \partial_x(a(x)\partial_x u(x,t)) + c(x)u(x,t) = f(x,t), \quad (x,t) \in Q, \\ u(0,t) = u(1,t) = 0, \quad \forall t \in [0,T], \\ \partial_t u(x,0) = v_0(x), \quad \forall x \in \Omega, \\ u(x,0) = u_0(x), \quad \forall x \in \Omega. \end{cases} \tag{1.1}$$

Here, $\Omega = [0,1]$, $Q = \Omega \times [0,T]$, $(u_0, v_0) \in H_a^1(\Omega) \times L^2(\Omega)$, where $T > 0$ is a fixed constant, and $f \in L^2(Q)$ represents the source term. Additionally, we assume that $a(.) \in C^1(\Omega)$ degenerates at a point x_0 within the spatial domain Ω, with $a(x) \leq 1$ for all $x \in \Omega$, and $c \in L^\infty(\Omega)$ is a positive function with $c(x_0) = 0$.

In recent years, inverse source problems related to wave equations have captured the interest of the scientific community as they provide a representative model for many applications mentioned above. Ahang et al. [41] investigated the inverse problem of recovering the trajectory $p(t)$ of the source of moving points within the limited-time window $[0,t]$ from the measured data $u(x,t)$. They also predicted the trajectory for the subsequent period $[t,T]$ based on the trajectory in the limited-time window $[0,t]$.

Xianli Lv et al. [33] addressed the inverse problem of determining the space-dependent source function and initial value for the time fractional nonhomogeneous diffusion-wave equation in a multi-dimensional context, using noisy final time measured data. They introduced a novel mollification regularization technique employing a bilateral exponential kernel to effectively handle the ill-posed nature of the problem. The paper also established error estimates, offering both a priori strategies and an a posteriori parameter selection rule for regularization. Through numerical experiments, the authors demonstrated the method's effectiveness and robustness in handling data perturbations.

El Badia et al. [2] tackled the determination of a single moving point source from boundary data for a three-dimensional wave equation.

Lesnic et al. [23] explored the problem of identifying an unknown dependent force function acting on a vibrating string from Cauchy measurements on the excess boundary. In the case of a one-dimensional wave equation, the boundary element method [22], combined with a regularized separate variable method, was used to solve the inverse problem of a space-dependent force function.

As discussed by Rundel [36] and others [35, 15, 9], determining $f = f(x,t)$ when considering its space and time dependence is a complex task that requires knowledge of the solution u.

Inverse source problem (ISP) In this study, our focus centers around the task of determining $h(x)$, which represents the spatial component of the source term $f(x,t)$ in problem (1.1). More precisely, we express $f(x,t)$ as the product $h(x)R(x,t)$. We will assume that h exhibits sufficient smoothness and remains independent of the time variable t, while $R(x,t) \in L^\infty(Q)$. Additionally, we posit the opportunity to incorporate further information for the inverse heat problems at the final time T:

$$u(x,T) = \tilde{u}(x) \quad \text{for all } x \text{ within } \Omega \tag{1.2}$$

Here, the provided observation data with inherent noise is denoted as $\tilde{u}(x) \in L^2(\Omega)$ and adheres to homogeneous Dirichlet boundary conditions. The challenge of identifying the source term will guide us towards the minimization of the functional denoted as J, formulated as follows:

$$\min_{h\in\mathcal{U}} J(h), \quad J(h) = \frac{1}{2}\|u(\cdot,T) - \tilde{u}\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\|h\|^2_{L^2(\Omega)}, \tag{1.3}$$

subject to u being the weak solution of the degenerate hyperbolic problem (1.1). $\tilde{u} \in L^2(\Omega)$ is the observation data with noise, $\varepsilon > 0$ is the regularization parameter, and $\mathcal{U}$ is the set of admissible sources defined as follows:

$$\mathcal{U} := \{h \in H^1(\Omega) : \|h\|_{H^1(\Omega)} \leq r, r > 0\}. \tag{1.4}$$

Evidently, $\mathcal{U}$ is a bounded, closed, and convex subset of $H^1(\Omega)$.

The main contribution of this paper is the presentation of numerical and theoretical results from the current study. We analyze various theoretical aspects associated with the problem under investigation, such as the existence and uniqueness of the direct solution, in parallel with the stability and regularization of the inverse problem. In other words, we approach the solution of the inverse source problem using a computational approach based on Tikhonov regularization. Based on the reformulation of the problem from the final data $u(T)$, two fundamental issues are discussed. First, we examine the existence and uniqueness of the specified source function. Second, we later address the issue of the stability of the optimization problem related to perturbations of the observed data.

A similar methodology has been previously utilized in scholarly works. For instance, in [10], the approach was employed to ascertain the initial condition of a degenerate parabolic equation in a two-dimensional space. Similarly, in [1], this methodology was adopted to identify the initial condition of a parabolic equation featuring a memory term. Furthermore, analogous techniques have been applied in related studies.

The structure of this paper is outlined as follows: The remainder of Section 1 is dedicated to addressing the analysis of both direct and inverse problems, focusing on aspects of existence and uniqueness. This is achieved through a comprehensive discussion on the continuity and differentiability of the functional J. In Section 2, formal proofs for the acquired results are provided. Moving forward to Section 3, we introduce the adjoint state technique, which plays a pivotal role in computing the gradient of J. Section 4 is dedicated to establishing the Lipschitz continuity of the gradient of the functional J. The "Thresholding" algorithm is presented in Section 5. Finally, Section 6 comprises a series of numerical simulations aimed at illustrating the alignment between our theoretical findings and empirical observations.

We consider the following Hilbert space:

$$H^1_a(\Omega) = \left\{\psi \in W^{1,2}_0(\Omega) \,:\, \sqrt{a}\psi_x \in L^2(\Omega)\right\}$$

endowed with the inner product

$$\langle u, v\rangle_{H_a^1} = \int_0^1 a u' v' \, dx + \int_0^1 uv \, dx.$$

Let the function $f(x,t) = h(x)R(x,t)$ for all $(x,t) \in Q$. The weak formulation of the problem (1.1) is:

$$\int_\Omega \partial_{tt} uv \, dx + \int_\Omega a(x)\partial_x u \partial_x v \, dx + \int_\Omega c(x) uv \, dx = \int_\Omega fv \, dx, \quad \forall v \in H_0^1(\Omega). \tag{1.5}$$

We have the following result:

Theorem 1. Let $v_0 \in L^2(\Omega)$ and $u_0 \in H_a^1(\Omega)$. Then the problem (1.1) has a unique weak solution such that

$$u \in L^2\left(0, T; H_a^1(\Omega)\right) \cap L^\infty\left(0, T; L^2(\Omega)\right), \quad \partial_t u \in L^2(0, T; L^2(\Omega)), \tag{1.6}$$

and we have the estimate

$$\begin{aligned} \sup_{t\in[0,T]} \|u(x,t)\|^2_{L^2(\Omega)} + \|\partial_t u(x,t)\|^2_{L^2(0,T;L^2(\Omega))} + \|\sqrt{a(x)}\partial_x u(x,t)\|^2_{L^2(0,T;L^2(\Omega))} \\ \leq C\left(\|u_0\|^2_{H_a^1(\Omega)} + \|v_0\|^2_{L^2(\Omega)} + \|h(x,t)\|^2_{L^2(\Omega)}\right). \end{aligned} \tag{1.7}$$

The constant C depends on Ω and T.

An elementary result to show the continuity of the functional J is the following lemma:

Lemma 1. Under the assumptions of Theorem 1, let u be the weak solution of (1.1) corresponding to a given source term h. Then, the evolution operator

$$F : L^2(\Omega) \to L^2\left(0, T; H_a^1(\Omega)\right) \cap L^\infty\left(0, T; L^2(\Omega)\right), \quad F(h) = u$$

is Lipschitz continuous.

An immediate consequence of Lemma 1 is the following proposition:

Proposition 1. Under the assumptions of Theorem 1, the functional J is continuous on $\mathcal{U}$, and there exists a unique minimizer $h^\star \in \mathcal{U}$, i.e.,

$$J(h^\star) = \min_{h\in\mathcal{U}} J(h).$$

To move on to the differentiability of J, we first present the following lemma:

Lemma 2. Let u be the weak solution of (1.1) with source term h. The evolution operator

$$F : L^2(\Omega) \to L^2\left(0, T; H_a^1(\Omega)\right) \cap L^\infty\left(0, T; L^2(\Omega)\right),$$

$F(h) = u$ is G-differentiable.

Expanding on Lemma (2) introduced earlier, we proceed to state the subsequent proposition, which is a direct consequence of the lemma's findings:

Proposition 2. Under the assumptions of Theorem 1, the functional J is G-differentiable on $\mathcal{U}$.

2 Proof of main results

In this section, we will demonstrate the results established in the previous section.

Proof of Theorem 1. The demonstration hinges on the framework employed to establish the theorem of existence and uniqueness in the context of the degenerate linear viscoelastic problem as outlined in [8]. We retrace the logical progression utilized therein to enhance clarity. To achieve this, we turn our attention to the subsequent non-degenerate wave problem. For any positive integer n, we consider the following non-degenerate wave problem:

$$\begin{cases} \partial_{tt}u^n - \partial_x\left(\left(a(x)+\dfrac{1}{n}\right)\partial_x u^n\right) + c(x)u^n = f(x,t), \quad \forall x \in Q, \\ u^n(0,t) = u^n(1,t) = 0, \quad \forall t \in [0;T], \\ u^n(x,0) = u_0(x), \quad \forall x \in \Omega, \\ \partial_t u^n(x,0) = v_0(x), \quad \forall x \in \Omega. \end{cases} \tag{2.1}$$

By the classical theory of wave equations (see [30]), the system (2.1) admits a unique weak solution u^n. Multiply the first equation of (2.1) by $\partial_t u^n$ and integrate it on Ω, we get

$$\begin{aligned} \int_\Omega \partial_{tt}u^n \partial_t u^n \, dx - \int_\Omega \partial_x\left(\left(a(x)+\frac{1}{n}\right)\partial_x u^n\right)\partial_t u^n \, dx \\ + \int_\Omega c(x)u^n\partial_t u^n dx = \int_\Omega f\partial_t u^n \, dx. \end{aligned} \tag{2.2}$$

Ω is independent of t, by the formula of green we obtain

$$\begin{aligned} \int_\Omega \partial_{tt}u^n\partial_t u^n dx + \int_\Omega \left(a(x)+\frac{1}{n}\right)\partial_x u^n \partial_{xt}u^n dx + \int_\Omega c(x)u^n\partial_t u^n dx \\ \leq \int_\Omega f\partial_t u^n dx. \end{aligned} \tag{2.3}$$

Which results in

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega (\partial_t u^n)^2 \, dx + \frac{1}{2}\frac{d}{dt}\int_\Omega \left(a(x)+\frac{1}{n}\right)(\partial_x u^n)^2 \, dx + \frac{1}{2}\frac{d}{dt}\int_\Omega c(x)(u^n)^2 \, dx \\ \leqslant \int_\Omega f\partial_t u^n \, dx. \end{aligned} \tag{2.4}$$

We integrate from 0 to t, with $t \in [0,T]$, which gives

$$\begin{aligned} \| \partial_t u^n(t) \|^2_{L^2(\Omega)} + \| \sqrt{\left(a(x)+\frac{1}{n}\right)}\partial_x u^n(t) \|^2_{L^2(\Omega)} + \int_\Omega c(x)(u^n(t))^2 dx \\ \leqslant \left[\| v_0 \|^2_{L^2(\Omega)} + \| \sqrt{\left(a(x)+\frac{1}{n}\right)}\partial_x u_0 \|^2_{L^2(\Omega)} + \| c\|_{L^\infty(\Omega)}\|u_0\|^2_{L^2(\Omega)}\right] \\ + \| f \|^2_{L^2(0,T;L^2(\Omega))} + \int_0^t \| \partial_t u^n(s) \|^2_{L^2(\Omega)} \, ds. \end{aligned} \tag{2.5}$$

We have $a(x) \leqslant 1 \ \forall x \in \Omega$, and $\dfrac{1}{n} < 1$, then

$$\sqrt{a(x) + \frac{1}{n}} \leqslant \sqrt{2} \quad \forall x \in \Omega. \tag{2.6}$$

We pose

$$C_1 = \left[\| v_0 \|^2_{L^2(\Omega)} + max(2, \|c\|_{L^\infty(\Omega)}) \| u_0 \|^2_{H^1(\Omega)}\right] + \| f \|^2_{L^2(0,T;L^2(\Omega))} . \tag{2.7}$$

Since

$$\int_\Omega c(x)(u^n(t))^2 dx \geqslant 0 \tag{2.8}$$

As a result

$$\begin{aligned} \| \partial_t u^n(t) \|^2_{L^2(\Omega)} + \| \sqrt{\left(a(x) + \frac{1}{n}\right)} \partial_x u^n(t) \|^2_{L^2(\Omega)} \\ \leqslant C_1 + \int_0^t \| \partial_t u^n(s) \|^2_{L^2(\Omega)} ds. \end{aligned} \tag{2.9}$$

We have

$$u^n(x,t) = \int_0^t \partial_t u^n(x,s) ds + u^n(x,0), \tag{2.10}$$

then

$$(u^n(x,t))^2 = \left(\int_0^t \partial_t u^n(x,s) ds\right)^2 + (u^n(x,0))^2 + 2u^n(x,0) \int_0^t \partial_t u^n(x,s) ds. \tag{2.11}$$

By the Hölder inequality , we arrive at

$$\int_0^t | \partial_t u^n(x,s) | ds \leqslant \sqrt{t} \left(\int_0^t (\partial_t u^n(x,s))^2 ds\right)^{\frac{1}{2}} . \tag{2.12}$$

From where,

$$\left(\int_0^t \partial_t u^n(x,s) ds\right)^2 \leqslant T \left(\int_0^t (\partial_t u^n(x,s))^2 ds\right). \tag{2.13}$$

Moreover, we have $2ab \leqslant a^2 + b^2$,

$$\begin{aligned} 2u^n(x,0) \int_0^t \partial_t u^n(x,s) ds &\leqslant (u^n(x,0))^2 + \left(\int_0^t \partial_t u^n(x,s) ds\right)^2 \\ &\leqslant (u^n(x,0))^2 + T \left(\int_0^t (\partial_t u^n(x,s))^2 ds\right). \end{aligned} \tag{2.14}$$

Using (2.11), (2.13) and (2.14) we conclude that

$$\| u^n(t) \|^2_{L^2(\Omega)} \leqslant 2 \| u_0 \|^2_{L^2(\Omega)} + 2T \int_0^t \| \partial_t u^n(s) \|^2_{L^2(\Omega)} ds. \tag{2.15}$$

Let

$$C_2 = C_1 + 2 \parallel u_0 \parallel^2_{L^2(\Omega)} \text{ and } M = 2T + 1, \tag{2.16}$$

and

$$y(t) = \parallel u^n(t) \parallel^2_{L^2(\Omega)} + \parallel \partial_t u^n(t) \parallel^2_{L^2(\Omega)} + \parallel \sqrt{\left(a(x) + \frac{1}{n}\right)} \partial_x u^n(t) \parallel^2_{L^2(\Omega)} . \tag{2.17}$$

Using (2.9) and (2.15) we obtain

$$y(t) \leqslant C_2 + M \int_0^t y(s) ds. \tag{2.18}$$

By Gronwall's Lemma

$$y(t) \leqslant C_2 \exp(MT) \qquad \forall t \in [0, T]. \tag{2.19}$$

Let $M_1 = C_2 \exp(MT)$. Then $\forall t \in [0, T]$ we have

$$\begin{aligned} &\parallel u^n(t) \parallel^2_{L^2(\Omega)} + \parallel \partial_t u^n(t) \parallel^2_{L^2(\Omega)} \\ &\qquad + \parallel \sqrt{\left(a(x) + \frac{1}{n}\right)} \partial_x u^n(t) \parallel^2_{L^2(\Omega)} \leqslant M_1. \end{aligned} \tag{2.20}$$

Therefore,

$$\parallel \partial_t u^n(t) \parallel^2_{L^2(\Omega)} \leqslant M_1, \ \forall t \in [0, T], \tag{2.21}$$

$$\parallel u^n(t) \parallel^2_{L^\infty(0,T;L^2(\Omega))} \leqslant M_1, \ \forall t \in [0, T], \tag{2.22}$$

$$\parallel \sqrt{a(x) + \frac{1}{n}} \partial_x u^n(t) \parallel^2_{L^2(\Omega)} \leqslant M_1, \ \forall t \in [0, T], \tag{2.23}$$

and there exists a sub-sequence $\{u^{n_j}\}$ of $\{u^n\}$ and a function

$$u \in L^\infty(0, T; L^2(\Omega)) \cap L^2 \left(0, T; H^1_a(\Omega)\right)$$

which satisfies $\partial_t u \in L^\infty(0, T; L^2(\Omega))$, and that when $n_j \to \infty$

$$\begin{aligned} &u^{n_j} \overset{*}{\rightharpoonup} u \qquad \text{weakly- * on } L^\infty(0, T; L^2(\Omega)), \\ &u^{n_j} \rightharpoonup u \qquad \text{weakly on } L^2(0, T; H^1_a(\Omega)), \\ &\partial_t u^{n_j} \rightharpoonup \partial_t u \qquad \text{weakly on } L^2(0, T; L^2(\Omega)), \\ &\sqrt{\left(a(x) + \frac{1}{n_j}\right)} \partial_x u^{n_j} \rightharpoonup \sqrt{a(x)} \partial_x u \qquad \text{weakly on } L^2(0, T; L^2(\Omega)). \end{aligned} \tag{2.24}$$

Since $\{u^{n_j}\}$ is the solution of system (2.1), we have $u^{n_j}(x,0) = u_0$ and $\partial_t u^{n_j}(x,0) = v_0$. Let's go back to the equation (2.4), by integration from 0 to T we get

$$\begin{aligned} &\frac{1}{2}\int_\Omega (\partial_t u^{n_j})^2(T)dx - \frac{1}{2}\int_\Omega (\partial_t u^{n_j})^2(0)dx + \frac{1}{2}\int_\Omega \left(a(x)+\frac{1}{n_j}\right)(\partial_x u^{n_j})^2(T)dx \\ &-\frac{1}{2}\int_\Omega \left(a(x)+\frac{1}{n_j}\right)(\partial_x u^{n_j})^2(0)dx + \frac{1}{2}\int_\Omega c(x)(u^{n_j})^2(T)dx - \frac{1}{2}\int_\Omega c(x)(u^{n_j})^2(0)dx \\ &\leqslant \int_0^T \int_\Omega f\partial_t u^{n_j} dxdt. \end{aligned} \tag{2.25}$$

Using (2.20), we get

$$\| \partial_t u^{n_j} \|^2_{L^2(0,T,L^2(\Omega))} \leqslant TM_1. \tag{2.26}$$

The weak formulation of (2.1) is

$$\begin{aligned} \int_\Omega \partial_{tt} u^{n_j} v\ dx + \int_\Omega \left(a(x)+\frac{1}{n_j}\right)\partial_x u^{n_j}\partial_x v\ dx + \int_\Omega c(x)u^{n_j} v\ dx \\ = \int_\Omega fv\ dx \qquad \forall v \in H_0^1(\Omega). \end{aligned} \tag{2.27}$$

If we take $v \in H_0^1(\Omega)$ such that $\| v \|_{H_0^1(\Omega)} \leqslant 1$, we obtain

$$\begin{aligned} < \partial_{tt} u^{n_j}, v >_{L^2(\Omega)} \leqslant \| \sqrt{a(x)+\frac{1}{n_j}}\partial_x u^{n_j} \|_{L^2(\Omega)} \| v \|_{H_0^1(\Omega)} \\ + \|c\|_{L^\infty(\Omega)} \|u^{n_j}\|_{L^2(\Omega)} \| v \|_{H_0^1(\Omega)} + \| f \|_{L^2(\Omega)} \| v \|_{H_0^1(\Omega)} \end{aligned} \tag{2.28}$$

$$\| \partial_{tt} u^{n_j} \|_{H^{-1}(\Omega)} \leqslant \| \sqrt{a(x)+\frac{1}{n_j}}\partial_x u^{n_j} \|_{L^2(\Omega)} + \|c\|_{L^\infty(\Omega)} \|u^{n_j}\|_{L^2(\Omega)} + \| f \|_{L^2(\Omega)}. \tag{2.29}$$

Using (2.20) we get

$$\| \partial_{tt} u^{n_j} \|_{L^2(0,T;H^{-1}(\Omega))} < \infty. \tag{2.30}$$

Hence

$$\partial_{tt} u^{n_j} \text{ is bounded on } L^2(0,T,H^{-1}(\Omega)). \tag{2.31}$$

We conclude that

$$
\begin{aligned}
u^{n_j} &\overset{*}{\rightharpoonup} u \qquad \text{weakly-}*\ \text{on} L^\infty(0,T;L^2(\Omega)),\\
u^{n_j} &\rightharpoonup u \qquad \text{weakly on } L^2(0,T;H^1_a(\Omega)),\\
\partial_t u^{n_j} &\rightharpoonup \partial_t u \qquad \text{weakly on } L^2(0,T;L^2(\Omega)),\\
\sqrt{\left(a(x)+\frac{1}{n_j}\right)}\partial_x u^{n_j} &\rightharpoonup \sqrt{a(x)}\partial_x u \qquad \text{weakly on } L^2(0,T;L^2(\Omega)),\\
c(x)u^{n_j} &\rightharpoonup c(x)u \qquad \text{weakly on } L^2(0,T;L^2(\Omega)),\\
\partial_{tt} u^{n_j} &\rightharpoonup \partial_{tt} u \qquad \text{weakly on } L^2(0,T;H^{-1}(\Omega)),
\end{aligned}
\tag{2.32}
$$

Passing to the weak limit

$$
\begin{aligned}
\int_\Omega \partial_{tt} uv dx + \int_\Omega a(x)\partial_x u \partial_x v dx + \int_\Omega c(x)uv dx \\
= \int_\Omega fv,\ \forall v \in H^1_0(\Omega),\quad a.e. t \in [0,T].
\end{aligned}
\tag{2.33}
$$

We get that u is the weak solution of (1.1)

Now, we prove the existence of the weak solution of (1.1) for each $(u_0, v_0) \in H^1_a(\Omega) \times L^2(\Omega)$ and $f \in L^2(0;T;L^2(\Omega))$. Let $\{u_0^m\}$, $\{v_0^m\}$ and $\{f^m\}$ of Cauchy sequences, respectively, such that for $m \longrightarrow \infty$, $u_0^m \longrightarrow u_0$ on $H^1_a(\Omega)$, $v_0^m \longrightarrow v_0$ in $L^2(\Omega)$ and $f^m \longrightarrow f$ on $L^2(0,T;L^2(\Omega))$.

Denote by u^m the solution of (1.1) associated with (u_0^m, v_0^m) and f^m, and u^n the solution of (1.1) associated with (u_0^n, v_0^n) and f^n.

We have the following variational problem, for all $v \in H^1_0(\Omega)$

$$
\left\{
\begin{array}{l}
\displaystyle\int_\Omega \partial_{tt}(u^n - u^m)v dx + \int_\Omega a\partial_x(u^n - u^m)\partial_x v dx + \int_\Omega c(x)(u^n - u^m)v dx = \int_\Omega (f^n - f^m)v\ dx,\\
(u^n - u^m)(x,t) = 0, \forall x \in \partial\Omega,\quad \forall t \in]0;T[,\\
(u^n - u^m)(x,0) = (u_0^n - u_0^m),\quad \forall x \in \Omega,\\
\partial_t(u^n - u^m)(x,0) = (v_0^n - v_0^m),\quad \forall x \in \Omega.
\end{array}
\right.
\tag{2.34}
$$

Similar to the equations (2.20) and (2.23), let $M = 2T + 1$, and

$$
\begin{aligned}
N_1 = (4 + \|c\|_{L^\infty}) exp(MT) \Big[&\| v_0^n - v_0^m \|^2_{L^2(\Omega)} \\
&+ \| u_0^n - u_0^m \|^2_{H^1(\Omega)} + \| f^n - f^m \|^2_{L^2(0,T;L^2(\Omega))} \Big].
\end{aligned}
$$

We obtain the following estimates:

$$
\| \partial_t u^n - \partial_t u^m \|^2_{L^2(0,T;L^2(\Omega))} \leqslant N_1, \tag{2.35}
$$

$$\| u^n - u^m \|^2_{L^\infty(0,T;L^2(\Omega))} \leqslant N_1, \tag{2.36}$$

$$\| u^n - u^m \|^2_{L^2(0,T;H^1_a(\Omega))} \leqslant N_1. \tag{2.37}$$

Then there exists $u \in L^2(0,T;H^1_a(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ and $\partial_t u \in L^2(0,T;L^2(\Omega))$, such that for $m \longrightarrow \infty$

$$\begin{gathered} u^m \longrightarrow u \text{ in } L^\infty(0,T;L^2(\Omega)), \text{ and } \partial_t u^m \longrightarrow \partial_t u \text{ in } L^2(0,T;L^2(\Omega)), \\ \text{and } u^m \longrightarrow u \text{ in } L^2(0,T;H^1_a(\Omega)). \end{gathered} \tag{2.38}$$

Now, we prove that the weak solution of the problem (1.1) is unique.
Let u_1 and u_2 be two weak solutions of the problem (1.1).
Let $Du = u_1 - u_2$, consequently Du satisfies

$$\begin{cases} \displaystyle\int_\Omega \partial_{tt} Duv\ dx + \int_\Omega a(x)\partial_x Du \partial_x v\ dx + \int_\Omega c(x) Duv dx = 0, \quad \forall v \in H^1_0(\Omega) \\ Du(x,t) = 0, \quad \forall x \in \partial\Omega \quad, \forall t \in]0;T[\\ Du(x,0) = 0, \quad \forall x \in \Omega \\ \partial_t Du(x,0) = 0, \quad \forall x \in \Omega. \end{cases} \tag{2.39}$$

In the same way to obtain the equation (2.20) we have

$$\| Du \|^2_{L^2(0,T;H^1_a(\Omega))} = 0. \tag{2.40}$$

Hence

$$u_1 = u_2, \quad a.e.\ t \in [0,T].$$

Returning to the estimate(1.7), we have from (2.21), (2.20) and (2.23), we find the following estimates :

$$\| \partial_t u^n(t) \|^2_{L^2(0,T;L^2(\Omega))} \leqslant TM_1,\ \forall t \in [0,T], \tag{2.41}$$

$$\| u^n(t) \|^2_{L^2(\Omega)} \leqslant M_1,\ \forall t \in [0,T], \tag{2.42}$$

$$\| \sqrt{a(x) + \frac{1}{n}} \partial_x u^n(t) \|^2_{L^2(0,T;L^2(\Omega))} \leqslant M_1 T,\ \forall t \in [0,T], \tag{2.43}$$

We have also

$$\begin{aligned} \|f\|^2_{L^2(0,T;L^2(\Omega))} &= \int_0^T \|h(x)R(x,t)\|^2_{L^2(\Omega)} dx \\ &\leq T\|R\|^2_{L^\infty(Q)} \|h\|^2_{L^2(\Omega)}. \end{aligned} \tag{2.44}$$

By combining the estimates (2.41),(2.42), (2.43), (2.44) and crossing the limit, we find the estimate sought Q.E.D.

Proof of the Lemma 1. Let the source term h be perturbed by a small amount δh such that $h+\delta h \in \mathcal{U}$. Consider $\delta u := u^\delta - u$, where u^δ is the weak solution of (1.1) with source term $h^\delta := h + \delta h$. Then $\delta u \in L^2\left(0,T; H_a^1(\Omega)\right) \cap L^\infty\left(0,T; L^2(\Omega)\right)$ satisfies the following variational problem, for all $v \in H_0^1(\Omega)$:

$$\begin{cases} \displaystyle\int_\Omega \partial_{tt}\delta u v \, dx + \int_\Omega a(x)\partial_x \delta u \partial_x v \, dx + \int_\Omega c(x)\delta u v dx = \int_\Omega \delta h(x) R(x,t) v dx, \\ \delta u(t,x) = 0, \qquad \forall t \in]0;T[\quad, \forall x \in \partial\Omega, \\ \delta u(t=0) = 0, \qquad \forall x \in \Omega, \\ \partial_t \delta u(t=0) = 0, \qquad \forall x \in \Omega. \end{cases} \tag{2.45}$$

Therefore, δu is the weak solution of (1.1) with $\delta u(t=0) = 0$ and $\partial_t \delta u(t=0) = 0$. We apply the estimation of Theorem 1, we get

$$\begin{aligned} \sup_{t\in[0,T]} \| \delta u(t) \|^2_{L^2(\Omega)} + \| \partial_t \delta u \|^2_{L^2(0,T;L^2(\Omega))} + \| \sqrt{a(x)}\partial_x \delta u \|^2_{L^2(0,T;L^2(\Omega))} \\ \leqslant C \| \delta h \|^2_{L^2(\Omega)} . \end{aligned} \tag{2.46}$$

Hence

$$\| \delta u \|^2_{L^\infty(0,T;L^2(\Omega))} \leqslant C \| \delta h \|^2_{L^2(\Omega)}, \tag{2.47}$$

and

$$\| \delta u \|^2_{L^2(0,T;H_a^1(\Omega))} \leqslant C \| \delta h \|^2_{L^2(\Omega)} . \tag{2.48}$$

This completes the proof Lemma 1. Q.E.D.

Proof of Proposition 1. Let u and u^δ respectively the weak solutions of (1.1) with source term h and h^δ.

We know that

$$\begin{aligned} J(h^\delta) - J(h) = \frac{1}{2}\left(\left\| u^\delta(T) - \tilde{u} \right\|^2_{L^2(\Omega)} - \| u(T) - \tilde{u} \|^2_{L^2(\Omega)} \right) \\ + \frac{\varepsilon}{2}\left(\left\| h^\delta \right\|^2_{L^2(\Omega)} - \| h \|^2_{L^2(\Omega)} \right). \end{aligned} \tag{2.49}$$

Then

$$\begin{aligned} & \left| J(h^\delta) - J(h) \right| \\ \leq \frac{1}{2} & \left| \left\| u^\delta(T) - \tilde{u} \right\|^2_{L^2(\Omega)} - \| u(T) - \tilde{u} \|^2_{L^2(\Omega)} \right| + \frac{\varepsilon}{2} \left| \left\| h^\delta \right\|^2_{L^2(\Omega)} - \| h \|^2_{L^2(\Omega)} \right|. \end{aligned} \tag{2.50}$$

We have

$$\begin{aligned}\left|\|h^\delta\|^2_{L^2(\Omega)} - \|h\|^2_{L^2(\Omega)}\right| &= \left|\int_\Omega \left((h^\delta)^2 - (h)^2\right) dx\right| \\ &= \left|\int_\Omega \left(h^\delta - h\right)\left(h^\delta + h\right) dx\right| \\ &= \left|\int_\Omega \delta h \,(2h + \delta h)\, dx\right| \\ &\leq \|\delta h\|_{L^2(\Omega)} \|2h + \delta h\|_{L^2(\Omega)}.\end{aligned} \tag{2.51}$$

Let's recall that $\forall a,\ b \in E$, with E is an normed space,

$$|\|a\| - \|b\|| \leq \|a - b\|.$$

Which gives

$$\begin{aligned}&\left|\ \left\|u^\delta(T) - \tilde{u}\right\|^2_{L^2(\Omega)} - \|u(T) - \tilde{u}\|^2_{L^2(\Omega)}\right| \\ &= \left|\left\|u^\delta(T) - \tilde{u}\right\|_{L^2(\Omega)} - \|u(T) - \tilde{u}\|_{L^2(\Omega)}\right| \left|\left\|u^\delta(T) - \tilde{u}\right\|_{L^2(\Omega)} + \|u(T) - \tilde{u}\|_{L^2(\Omega)}\right| \\ &\leq \left(\left\|u^\delta(T) - \tilde{u}\right\|_{L^2(\Omega)} + \|u(T) - \tilde{u}\|_{L^2(\Omega)}\right) \left\|u^\delta(T) - u(T)\right\|_{L^2(\Omega)} \\ &\leq \left(\left\|u(T) - \tilde{u} + u^\delta(T) - u(T)\right\|_{L^2(\Omega)} + \|u(T) - \tilde{u}\|_{L^2(\Omega)}\right) \left\|u^\delta(T) - u(T)\right\|_{L^2(\Omega)} \\ &\leq \left(\left\|u^\delta(T) - u(T)\right\|_{L^2(\Omega)} + 2\|u(T) - \tilde{u}\|_{L^2(\Omega)}\right) \left\|u^\delta(T) - u(T)\right\|_{L^2(\Omega)}.\end{aligned} \tag{2.52}$$

Since the evolution function

$$\varphi : u_0 \in L^2(\Omega) \longrightarrow u \in C\left([0,T]; L^2(\Omega)\right) \cap L^2\left(0,T; H^1_0(\Omega)\right),$$

is Lipschitz continuous (Lemma 1), we deduce: there is a constant $C > 0$ such that

$$\left\|u^\delta(T) - u(T)\right\|_{L^2(\Omega)} \leq C\,\|\delta h\|_{L^2(\Omega)}. \tag{2.53}$$

Therefore

$$\begin{aligned}\left|J(h^\delta) - J(h)\right| &\leq \frac{\varepsilon}{2}\,\|2h + \delta h\|_{L^2(\Omega)}\,\|\delta h\|_{L^2(\Omega)} \\ &\quad + \frac{\sqrt{C}}{2}\left(\sqrt{C}\,\|\delta h\|_{L^2(\Omega)} + 2\,\|u(T) - \tilde{u}\|_{L^2(\Omega)}\right)\|\delta h\|_{L^2(\Omega)} \\ &\leq \left(\frac{\varepsilon}{2}\,\|2h + \delta h\|_{L^2(\Omega)} + \frac{\sqrt{C}}{2}\left(\sqrt{C}\,\|\delta h\|_{L^2(\Omega)} + 2\,\|u(T) - \tilde{u}\|_{L^2(\Omega)}\right)\right)\|\delta h\|_{L^2(\Omega)}.\end{aligned} \tag{2.54}$$

When h^δ tends to h, δh tends to 0, then the right-hand side of the inequality tends to 0. Accordingly, J is continuous, on the compact $\mathcal{U}$, then there exists a unique minimizer $h^\star \in \mathcal{U}$ for J. Q.E.D.

Proof of Lemma 2. Let δh be a small amount such that $h+\delta h \in \mathcal{U}$, we define the function

$$F'(h) : \delta h \in \mathcal{U} \to \delta u, \tag{2.55}$$

where δu is the solution of the following variational problem, for all $v \in H_0^1(\Omega)$:

$$\begin{cases} \int_\Omega \partial_{tt}\delta u v \, dx + \int_\Omega a(x)\partial_x \delta u \partial_x v \, dx + \int_\Omega c(x)\delta u v dx = \int_\Omega \delta h(x) R(x,t) \, dx, \\ \delta u(0,t) = \delta u(1,t) = 0, \quad \forall t \in]0;T[, \\ \delta u(x,0) = 0, \quad \forall x \in \Omega, \\ \partial_t \delta u(x,0) = 0, \quad \forall x \in \Omega. \end{cases} \tag{2.56}$$

We pose

$$\varphi(b) = F(b+\delta b) - F(b) - F'(b)\delta b. \tag{2.57}$$

We want to show that

$$\varphi(b) = o(\delta b). \tag{2.58}$$

Easily, we can verify that φ is a solution of the following variational problem, for all $v \in H_0^1(\Omega)$:

$$\begin{cases} \int_\Omega \partial_{tt}\varphi v \, dx + \int_\Omega a(x)\partial_x \varphi \partial_x v \, dx + \int_\Omega c(x)\varphi v \, dx = \int_\Omega (\delta h - (\delta h)^2) R(x,t) v \, dx, \\ \varphi(0,t) = \varphi(1,t) = 0, \quad \forall t \in]0;T[, \\ \varphi(x,0) = 0, \quad \forall x \in \Omega, \\ \partial_t \varphi(x,0) = 0, \quad \forall x \in \Omega. \end{cases} \tag{2.59}$$

In the same way used in the proof of continuity, we deduce that

$$\| \varphi \|^2_{L^\infty(0,T;L^2(\Omega))} \leqslant C \left\| \delta h - (\delta h)^2 \right\|^2_{L^2(\Omega)}, \tag{2.60}$$

and

$$\| \varphi \|^2_{L^2(0,T;H_a^1(\Omega))} \leqslant C \left\| \delta h - (\delta h)^2 \right\|^2_{L^2(\Omega)}. \tag{2.61}$$

This completes the proof Lemma 2. Q.E.D.

Proof of Proposition 2. Let δh be a small amount such that $h + \delta h \in \mathcal{U}$ and s a real. Consider $u^\gamma := u(h + s\delta h)$ and $u := u(h)$ are respectively the solutions of problem (1.1) with source term $h^\gamma := h + s\delta h$ and h.

$$\begin{aligned} \left| \frac{J(h^\gamma) - J(h)}{s} \right| &= \left| \frac{1}{2s} \|u^\gamma(T) - \tilde{u}\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2s} \|h^\gamma\|^2_{L^2(\Omega)} \right. \\ &\qquad \left. - \frac{1}{2s} \|u(T) - \tilde{u}\|^2_{L^2(\Omega)} - \frac{\varepsilon}{2s} \|h\|^2_{L^2(\Omega)} \right| \\ &\leq \frac{1}{2s} \left| \|u^\gamma(T) - \tilde{u}\|^2_{L^2(\Omega)} - \|u(T) - \tilde{u}\|^2_{L^2(\Omega)} \right| \\ &\qquad + \frac{\varepsilon}{2s} \left| \|h^\gamma\|^2_{L^2(\Omega)} - \|h\|^2_{L^2(\Omega)} \right| \\ &\leq \frac{1}{2s} \left| \|u^\gamma(T) - \tilde{u}\|_{L^2(\Omega)} - \|u(T) - \tilde{u}\|_{L^2(\Omega)} \right| \left| \|u^\gamma(T) - \tilde{u}\|_{L^2(\Omega)} \right. \\ &\qquad \left. + \|u(T) - \tilde{u}\|_{L^2(\Omega)} \right| + \frac{\varepsilon}{2s} \left| \|h^\gamma\|^2_{L^2(\Omega)} - \|h\|^2_{L^2(\Omega)} \right|. \end{aligned} \tag{2.62}$$

Using the same method as for the estimates 2.51 and 2.52, we get

$$\left|\frac{J(h^\gamma)-J(h)}{s}\right| \leq \frac{1}{2s}\|u^\gamma(T)-u(T)\|_{L^2(\Omega)}\Big|\|u^\gamma(T)-u(T)\|_{L^2(\Omega)} \\ +2\|u(T)-\tilde{u}\|_{L^2(\Omega)}\Big| + \frac{\varepsilon}{2s}\|s\delta h\|_{L^2(\Omega)}\|2h+s\delta h\|_{L^2(\Omega)}. \tag{2.63}$$

When s tends towards 0, the term $\frac{1}{2s}\|u^\gamma(T)-u(T)\|_{L^2(\Omega)}$ tends to $\frac{1}{2}\|\varphi\|_{L^2(\Omega)}$ and using the inequalities 2.60 and 2.61, we obtain:

$$\lim_{s\to 0}\left|\frac{J(h^\gamma)-J(h)}{s}\right| \leq \frac{\sqrt{C}}{2}\|\delta h-(\delta h)^2\|_{L^2(\Omega)}\left(\sqrt{C}\|\delta h-(\delta h)^2\|_{L^2(\Omega)} \\ +2\|u(T)-\tilde{u}\|_{L^2(\Omega)}\right)+\varepsilon\|\delta h\|_{L^2(\Omega)}\|h\|_{L^2(\Omega)}. \tag{2.64}$$

When δh tends to 0, we obtain the differentiability of J. Q.E.D.

3 The adjoint state method

In this section, we are going to compute the gradient of J with the adjoint state method. We define the Gâteaux derivative of u at h in the direction $k \in L^2(\Omega)$, by

$$\hat{u} = \lim_{s\to 0}\frac{u(h+sk)-u(h)}{s}, \tag{3.1}$$

$u(h+sk)$ is the solution of (1.1) with the term source $h+sk$, and $u(h)$ is the solution of (1.1) with the term source h.
We calculate the Gâteaux derivative of (1.1) at h in the direction $k \in L^2(\Omega)$, and we obtain the tangent linear model

$$\begin{gathered} \partial_t^2\hat{u} - \partial_x(a(x)\partial_x\hat{u}) + c(x)\hat{u} = k(x)R(x,t), \\ \hat{u}(0,t) = \hat{u}(1,t) = 0, \quad \forall t \in [0;T], \\ \hat{u}(x,0) = 0, \quad \forall x \in [0;1], \\ \partial_t\hat{u}(x,0) = 0, \quad \forall x \in [0;1]. \end{gathered} \tag{3.2}$$

We introduce the adjoint variable P, and we integrate

$$\int_0^1\int_0^T \partial_{tt}\hat{u}P\,dtdx - \int_0^1\int_0^T \partial_x(a(x)\partial_x\hat{u})P\,dtdx + \int_0^1\int_0^T c(x)\hat{u}P\,dtdx = \int_0^1\int_0^T k(x)R(x,t)P\,dtdx. \tag{3.3}$$

We pose $P(x=0) = P(x=1) = 0$. Since $\hat{u}(0,t) = \hat{u}(1,t) = 0\ \forall t \in [0,T]$, then $\partial_t\hat{u}(0,t) = \partial_t\hat{u}(1,t) = 0\ \forall t \in [0,T]$.
With $P(T) = 0$, we have

$$\int_0^1\int_0^T \partial_{tt}\hat{u}P\,dtdx = -\int_0^1 \partial_t P(x,T)\hat{u}(x,T)dx + \int_0^1\int_0^T \hat{u}\partial_{tt}P\,dtdx, \tag{3.4}$$

and

$$\int_0^1\int_0^T \partial_x(a(x)\partial_x\hat{u}(x))P\,dtdx = \int_0^1\int_0^T \partial_x(a(x)\partial_x P)\hat{u}(x)\,dtdx. \tag{3.5}$$

Then (3.3) becomes

$$
\begin{aligned}
&-\langle \partial_t P(T), \hat{u}(T)\rangle_{L^2(\Omega)} + \int_0^1\int_0^T \partial_{tt} P\hat{u} - \int_0^1\int_0^T \partial_x(a(x)\partial_x P)\hat{u} \\
&\quad + \int_0^1\int_0^T c(x)P\hat{u} = \left\langle k, \int_0^T R(x,t)P \right\rangle_{L^2(\Omega)},
\end{aligned}
\tag{3.6}
$$

which gives

$$
\begin{gathered}
\int_0^T \langle \partial_{tt}P - \partial_x(a(x)\partial_x P) + c(x)P, \hat{u}\rangle_{L^2(\Omega)} - \langle \partial_t P(T), \hat{u}(T)\rangle_{L^2(\Omega)} = \left\langle k, \int_0^T R(x,t)P \right\rangle_{L^2(\Omega)}, \\
P(x=0) = P(x=1) = 0,\ P(T) = 0.
\end{gathered}
\tag{3.7}
$$

The Gâteaux derivative of the functional

$$
J(h) = \frac{\varepsilon}{2}\,\|(h\|^2_{L^2(\Omega)} + \frac{1}{2}\,\|u(\cdot,T) - \tilde{u})\|^2_{L^2(\Omega)},
$$

at h in the direction $k \in L^2(\Omega)$ is given by

$$
\hat{J}(h) = \lim_{s\to 0}\frac{J(h+sk) - J(h)}{s}.
$$

After some calculations, we arrive at

$$
\hat{J}(h) = \langle \varepsilon h, k\rangle_{L^2(\Omega)} + \langle u(T) - \tilde{u}, \hat{u}(T)\rangle_{L^2(\Omega)}. \tag{3.8}
$$

The adjoint problem is

$$
\begin{cases}
\partial_{tt}P - \partial_x(a(x)\partial_x P) + c(x)P = 0, & (x,t) \in Q, \\
P(T) = 0,\ \partial_t P(T) = u(T) - \tilde{u}, & x \in \Omega, \\
P(x=0) = P(x=1) = 0, & t \in [0,T].
\end{cases}
\tag{3.9}
$$

Therefore, the gradient of J is given by

$$
\frac{\partial J}{\partial h} = \varepsilon h - \int_0^T R(x,t)P dt. \tag{3.10}
$$

This suggests a characterization of the solution to the minimization prob 1.3.

To calculate the gradient of J, we solve two problems: the direct problem (1.1), and the adjoint problem (3.9) with the change of variable $t \longleftrightarrow T - t$

4 Lipschitz continuity of the gradient

To ensure the convergence of the descent method, we present the following result

Proposition 3. Let h and δh, such that $h + \delta h \in \mathcal{U}$, Then ∇J is Lipschitz continuous

$$
\|\ \nabla J(h+\delta h) - \nabla J(h)\ \|_{L^2(\Omega)} \leqslant L_1 \ \|\ \delta h\ \|_{L^2(\Omega)}, \tag{4.1}
$$

with the Lipschitz constant $L_1 > 0$.

Proof of the Proposition 3. In the section 3 we have $\nabla J(h) = \varepsilon h - \int_0^T R(x,t)P_1 dt$ with P_1 is the solution of the adjoint model (with change of variable $t_j \longleftrightarrow T - t_j$). Let $\nabla J(h+\delta h) = \varepsilon(h+\delta h) - \int_0^T R(x,t)P_2 dt$. We consider $\delta P = P_2 - P_1$, we easily verify that δP is the solution of the problem

$$\begin{cases} \partial_{tt}\delta P - \partial_x(a(x)\partial_x \delta P + c(x)\delta P = 0, & (x,t) \in Q, \\ \delta P(T) = 0,\ \partial_t \delta P(T) = \delta u(T), & x \in \Omega, \\ \delta P(x=0) = \delta P(x=1) = 0, & t \in [0,T]. \end{cases} \tag{4.2}$$

Applying the estimates of Theorem 1, there exists a constant C such that

$$\sup_{t\in[0,T]} \| \delta P(t) \|^2_{L^2(\Omega)} + \| \partial_t \delta P \|^2_{L^2(0,T;L^2(\Omega))} \leqslant C \| \delta u(T) \|^2_{L^2(\Omega)} . \tag{4.3}$$

We have demonstrated the Lipschitz continuity of the evolution operator $F : h \longrightarrow u$. In particular, it has been shown that

$$\| \delta u \|^2_{L^\infty(0,T;L^2(\Omega))} \leqslant C \| \delta h \|^2_{L^2(\Omega)} . \tag{4.4}$$

Hence

$$\sup_{t\in[0,T]} \| \delta P(t) \|^2_{L^2(\Omega)} + \| \partial_t(t) \delta P \|^2_{L^2(0,T;L^2(\Omega))} \leqslant C^2 \| \delta h \|^2_{L^2(\Omega)} . \tag{4.5}$$

Moreover

$\nabla J(h+\delta h) - \nabla J(h) = \varepsilon \delta h - \int_0^T R(x,t)\delta P dt$. Which gives

$$\| \nabla J(h+\delta h) - \nabla J(h) \|^2_{L^2(\Omega)} \leqslant M \| \delta h \|^2_{L^2(\Omega)} . \tag{4.6}$$

With $L_1 = \left(C\sqrt{T} \| R \|_{L^\infty(Q)} + \varepsilon \right)^2$. This completes the proof of Proposition 3. Q.E.D.

5 "Thresholding" algorithm

Here we present the first algorithm, called "Thresholding", which is proposed for situations where no reference is available.

Proposition 4. A function $f^* \in \mathcal{U}$ minimizes the functional J only if it satisfies the following variational equation

$$\varepsilon f^* - \int_0^T R(x,t)P(f^*)dt = 0, \tag{5.1}$$

where, $P(f^*)$ solves the problem (3.9) with the coefficient f^*.

Adding μf^* ($\mu > 0$) to both sides of (5.1) and rearranging in view of the iteration, we are led to the iterative thresholding algorithm

$$f_{k+1} = \frac{\mu}{(\mu+\varepsilon)} f_k + \frac{1}{(\mu+\varepsilon)} \int_0^T R(x,t)P(f_k)dt, \quad k \in (0,1,...). \tag{5.2}$$

where $\mu > 0$ is a tuning parameter for the convergence. Similarly to [31], it follows from the general theory stated in [24] that it suffices to choose

$$\mu \geqslant \| F \|_{op}, \quad \text{with} \quad \begin{array}{r} F : L^2(\Omega) \longrightarrow L^2(\Omega) \\ f \longrightarrow u(\cdot, T). \end{array} \tag{5.3}$$

Here,

$$\|F\|_{op} = \sup_f \left(\frac{\|F(f)\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}} \right).$$

Algorithm.

1- $k = 0$

2- Choose f_k.

3- Calculate u_k *solution of* (1.1) *with the source term* f_k.

4- Calculate $P(f_k)$ *solution of* (3.9).

5- Calculate f_{k+1} *by iteration* (5.2).

6- If $\rho \leqslant \dfrac{\| f_{k+1} - f_k \|_{L^2(\Omega)}}{\| f_k \|_{L^2(\Omega)}}$ *then*

$k \longleftarrow k+1$ *and go back to step 3.*

If not, stop.

With ρ is a small precision given.

6 Numerical experiences

In this section, we will determine $h(x)$, source term part of (1.1), in both cases: the first, is when we don't have a reference value $\bar{h}(x)$ of $h(x)$. In this case, we will use the iterative "Thresholding" algorithm mentioned above.

In the second case, we will assume that we have $\bar{h}(x)$, to solve this inverse problem, we apply the descent algorithm

The main steps for descent method at each iteration are the following:

- Calculate u^k solution of (1.1) with source term h^k.
- Calculate P^k solution of the adjoint problem.
- Calculate the descent direction $d_k = -\nabla J(h^k)$.
- Find $t_k = \text{argmin}_{t>0} J(h^k + td_k)$.
- Update the variable $h^{k+1} = h^k + t_k d_k$.

The algorithm ends when $\mid J(h^k) \mid < \mu$, where μ is a given small precision.

t_k is chosen by the inaccurate linear search by the Armijo-Goldstein rule as follows:

let $\alpha_i, \beta \in [0, 1[$ and $\alpha > 0$

if $J(h^k + \alpha_i d_k) \leqslant J(h^k) + \beta \alpha_i d_k^T d_k$

$t_k = \alpha_i$ and stop

if not

$\alpha_i = \alpha \alpha_i$.

For all the tests we take $x_0 = 0.5, a(x) = (x - x_0)^2$ and $c(x) = (x - x_0)^2$.

6.1 Case without reference of the source term

We want to solve the minimization problem (1.3). But in practice, the data $\tilde{u}$ is noisy by measurement errors, we will assume that $\tilde{u} = u^{exact}(T) + e^{obs}$. Let $err^{obs} = \dfrac{\| e^{obs} \|_2}{\| u^{exact}(T) \|_2}$ and $err = \dfrac{\| h^{exact} - h \|_2}{\| h^{exact} \|_2}$, successively the observation error percentage and the reconstruction error, with h^{exact} is the exact source term to look for, and h is the constructed.

We will take the same examples as those used by [26], and we study the impact of err^{obs} on the construction of the source term.

For the discretization of direct and adjoint problems, we use the method θ-schema in time.

For all the tests we take $\mu = 40$, $\rho = 10^{-3}$, and $\varepsilon = 10^{-4}$.

- **Case** $h^{exact}(x) = cos(\pi x) + 2$

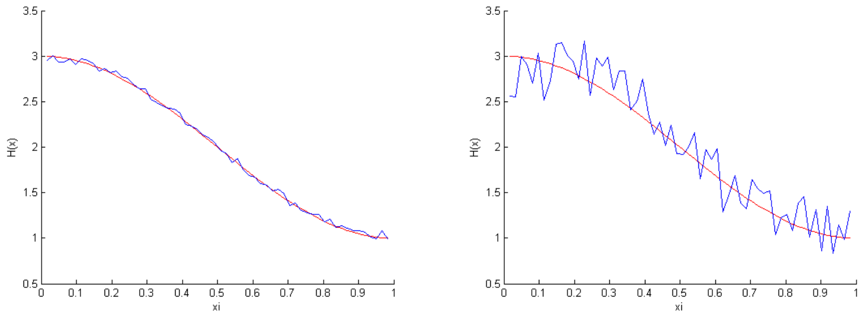


Figure 01. Test with $err^{obs} = 1\%$ (left), and $err^{obs} = 5\%$ (right).

- **Case** $h^{exact}(x) = 4 - e^{x}$

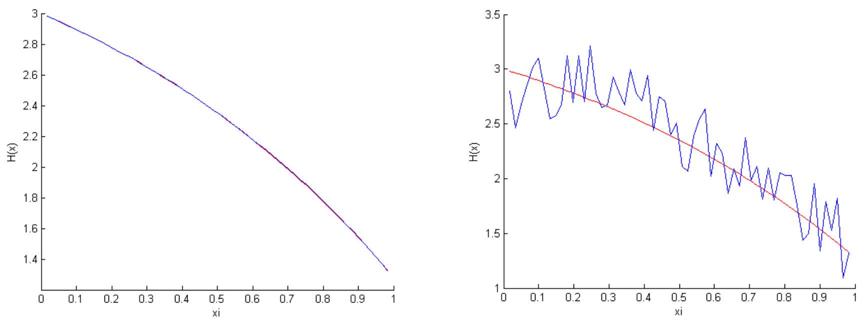


Figure 02. Test with $err^{obs} = 1\%$ (left), and $err^{obs} = 5\%$ (right).

The tests (Figures 1 - 2) show that the "Thresholding" algorithm is sensitive to the observation error err^{obs}.

6.2 Case with reference value of source term

In this case, we consider the following minimization problem:

$$\min_{h\in\mathcal{U}} J_R(h), \quad J_R(h) = \frac{1}{2}\|u(\cdot,T) - \tilde{u}\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\left\|h - \bar{h}(x)\right\|^2_{L^2(\Omega)}, \tag{6.1}$$

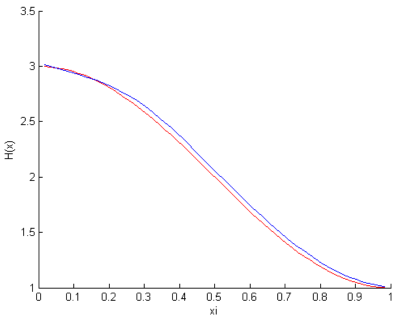
subject to u the weak solution of the degenerate hyperbolic problem (1.1), $\varepsilon > 0$ the regularization parameter.
The continuity and derivability of the functional J_R this deduces from Lemma 1 and Lemma 2. By the adjoint method, the gradient of J_R as follows

$$\frac{\partial J}{\partial h} = \varepsilon(h - \bar{h}) - \int_0^T R(x,t)P dt. \tag{6.2}$$

With P the solution of adjoint problem (3.9). To ensure the convergence of the gradient-type descent method, we have shown the Lipschitz continuity of ∇J.
We consider that the data $\bar{h}$ and $\tilde{u}$ are noisy by measurement errors $\bar{h} = h^{exact} + e$ and $\tilde{u} = u^{exact}(T) + e^{obs}$. Let $err = \dfrac{\| e \|_2}{\| h^{exact} \|_2}$ and $err^{obs} = \dfrac{\| e^{obs} \|_2}{\| u^{exact}(T) \|_2}$. Now, we realise two tests: in the first, we suppose that $err^{obs} = 0$, and we study the impact of err on the source term construction. In the second test, we set $err = 0$, and we study the impact of err^{obs} on the construction of source term.

Impact of err on source term construction

Case $h^{exact}(x) = cos(\pi x) + 2$.



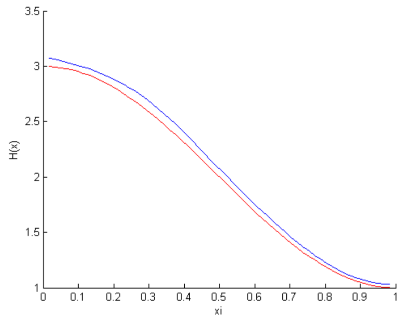


Figure 3. Test with $err = 1\%$ (left), and $err = 3\%$ (right). we can't rebuild the true state. But the reconstructed term source starts to get near to the true state

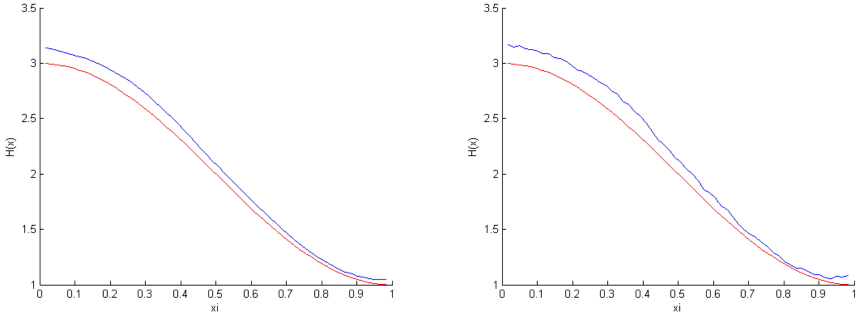


Figure 4. Test with $err = 5\%$ (left), and $err = 10\%$ (right).

Case $h^{exact}(x) = 4 - e^x$.

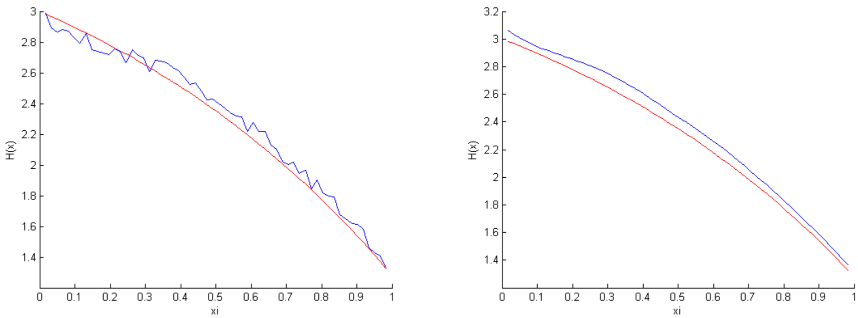


Figure 5. Test with $err = 1\%$ (left), and $err = 3\%$ (right).

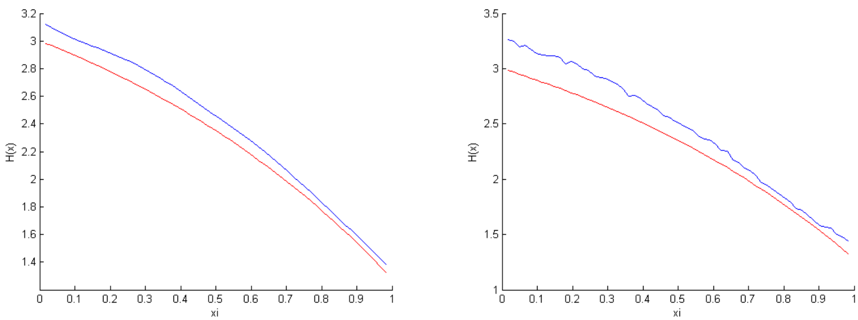


Figure 6. Test with $err = 5\%$ (left), and $err = 10\%$ (right).

Impact of err^{obs} on source term construction
Case $h^{exact}(x) = 4 - e^x$.

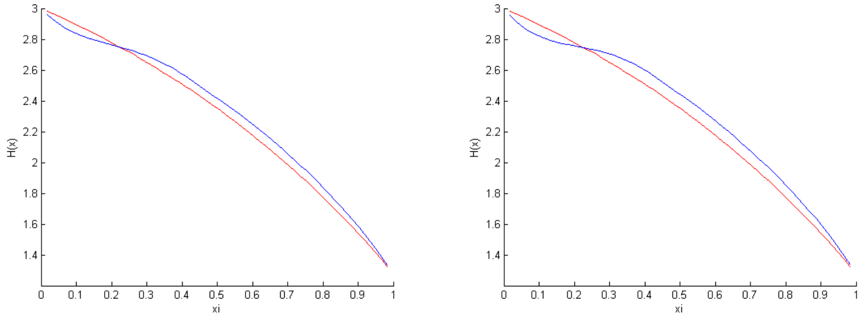


Figure 7. Test with $err^{obs} = 1\%$ (left), and $err^{obs} = 3\%$ (right).

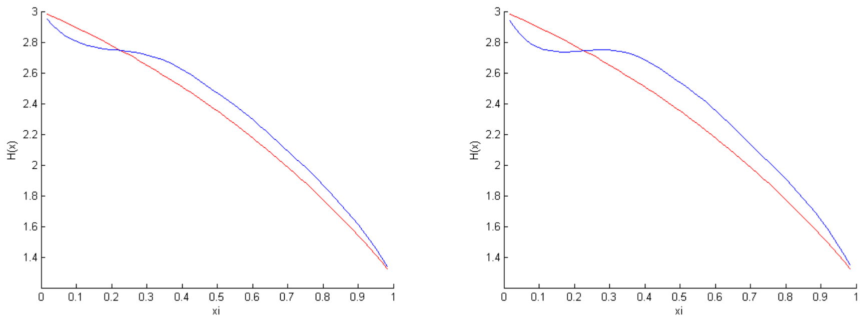


Figure 8. Test with $err^{obs} = 5\%$ (left), and $err^{obs} = 10\%$ (right).

Case $h^{exact}(x) = cos(\pi x) + 2$.

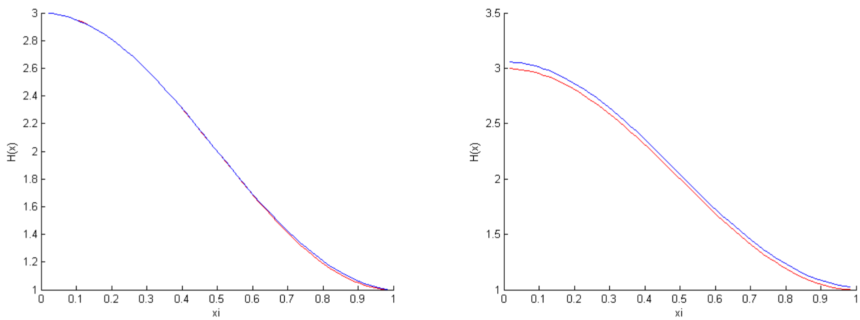


Figure 9. Test with $err^{obs} = 1\%$ (left), and $err^{obs} = 3\%$ (right).

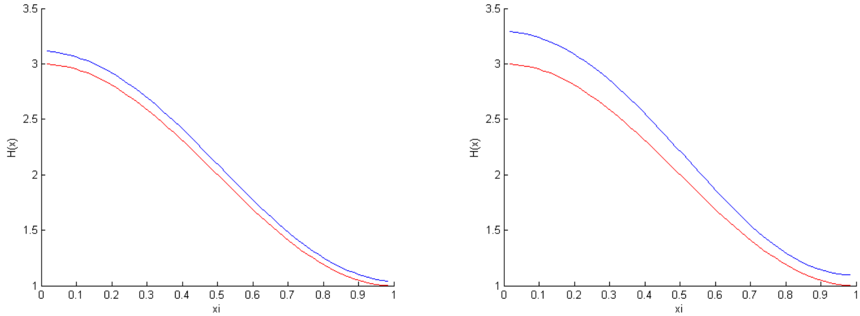


Figure 10. Test with $err^{obs} = 5\%$ (left), and $err^{obs} = 10\%$ (right).

The tests (Figures 3 to 10) show that the proposed algorithm is uniformly stable to observation noises. We observe that when the error exceeds 3% we cannot construct the source term.

7 Conclusion

In this work, we have presented a regularization method applied to determine the source term from final measurements for a degenerate hyperbolic problem occurring within the spatial domain. In the numerical section, we introduced two algorithms: a descent algorithm in cases where we have a reference for the source term, which we demonstrated to be resilient to observation noise. In the other scenario, we employed a "Thresholding" algorithm, which proved to be sensitive to observation noise, and we also noted that the execution time was considerably longer compared to when we have a reference for the source term.

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