Generalized Wintgen ideal Legendrian submanifolds of generalized Sasakian-space forms

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Abstract

The main idea of this article is to study the generalized Wintgen ideal Legendrian submanifolds of generalized Sasakian-space-forms. Also, we characterize generalized Wintgen ideal Legendrian submanifolds based on pseudo parallel and Ricci generalized pseudo parallel concerning Levi-Civita connection as well as the Schouten-Van Kampen and generalized Tanaka-Webster connections of generalized Sasakian-space-forms.

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1 Introduction

Wintgen [35] established the inequality $K \leq \|\Omega\|^2 - |K^{\perp}|$ between Gauss curvature K, the squared mean curvature $\|\Omega\|^2$ and normal curvature $K^{\perp}$ of any surface M^2 in E^4 and also shown that the equality holds if the ellipse of curvature of M^2 in E^4 is a circle. Later in 1999, De Smet et al. [11] have given the conjecture on Wintgen inequality for any submanifold in real space form

$$\rho \leq \|\Omega\|^2 - \rho^{\perp} + c, \tag{1.1}$$

where ρ is normalized scalar curvature and $\rho^{\perp}$ is normalized normal scalar curvature. They also proved this conjecture on a submanifold of arbitrary dimension and codimension 2 in real space form. Thereafter Choi and Lu [10] proved this inequality of any 3-dimensional submanifold and any codimension of real space form. In 2008, Ge and Tang [15] and in 2011, Lu [18] independently proved Wintgen inequality on submanifold of arbitrary dimension and codimension of real space form. Many authors studied Wintgen inequality of certain submanifold of different space forms, see [2, 13, 14, 19, 20, 21]. Chen [8] made a detailed survey of the recent results of Wintgen inequality. If the equality case of Wintgen inequality holds on a submanifold then such submanifold is said to be a Wintgen ideal submanifold. Several authors studied this submanifold and their geometric properties, such as [9, 10, 18, 19, 20].
In these context, Deszcz et al. [12] studied hypersurfaces in 4-dimensional space of constant curvature satisfying the condition

$$(\bar{R}(E_1, E_2) \cdot h)(E_3, E_4) = L_h Q(g, h)(E_3, E_4; E_1, E_2), \tag{1.2}$$

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$$(\bar{R}(E_1, E_2) \cdot h)(E_3, E_4) = L_S Q(S, h)(E_3, E_4; E_1, E_2), \tag{1.3}$$

where $\bar{R}$ is defined in (2.6).
Later Asperti et al. [3, 4] studied submanifold satisfying (1.2) and (1.3) in space forms. They named such submanifolds as pseudo parallel and Ricci generalized pseudo parallel respectively. Moreover, pseudo parallel contact CR-submanifolds were studied in [17]. Several authors studied pseudo parallel and Ricci pseudo parallel submanifolds of generalized Sasakian-space-forms [23, 32].
In 2008, Petrović-Torgašev and Verstraelen [26] have studied Deszcz symmetries of Wintgen ideal submanifolds of real space forms. Recently Šebeković et al. [28] studied pseudosymmetry properties of generalized Wintgen ideal Legendrian submanifold of Sasakian-space-form.
The Schouten-Van Kampen connection was introduced to study non-holomorphic manifolds [27]. Solov'ev [29, 30, 31] has investigated hyperdistributions in Riemannian manifolds using the Schouten-van Kampen connection. In 2006, Schouten-Van Kampen connection was studied on foliated manifolds by Bejancu [7]. Recently Olszak [24] studied such connection on almost(para) contact metric structure. Here we denote such connection by $\tilde{\bar{\nabla}}$.

The Tanaka-Webster connection [33, 36] was defined on a non-degenerate pseudo-Hermitian CR-manifold. In 1989, Tanno [34] defined generalized Tanaka-Webster connection for contact metric manifolds. Later Zamkovoy [38] defined generalized Tanaka-Webster connection for paracontact metric manifolds. Several authors studied Contact manifolds with generalized Tanaka-Webster connection [22, 25]. Here we denote such connection by $\overset{*}{\bar{\nabla}}$.
In this paper we have studied pseudo parallel and Ricci generalized pseudo parallel on generalized Wintgen ideal Legendrian submanifold of generalized Sasakian-space-form with respect to Levi-Civita connection, Schouten-Van Kampen connection and generalized Tanaka-Webster connection.

Remark 1.1. Throughout the paper, we use acronym "GWIL submanifold" for generalized Wintgen ideal Legendrian submanifold.

2 Preliminaries

An odd dimensional smooth manifold $\bar{M}^{2d+1}$ is said to be an almost contact metric manifold if the following holds [6]:

$$\varphi^2 E_1 = -E_1 + \eta(E_1)\chi, \quad \varphi\chi = 0, \tag{2.1}$$

$$g(E_1, \chi) = \eta(E_1), \quad \varphi \circ \eta = 0, \tag{2.2}$$

$$g(\varphi E_1, \varphi E_2) = g(E_1, E_2) - \eta(E_1)\eta(E_2), \tag{2.3}$$

where E_1, E_2 are the vector fields, φ is a tensor of type $(1, 1)$, χ is a vector field, η is an 1-form and g is a Riemannian metric on $\bar{M}$.
$\bar{M}^{2d+1}(\varphi, \chi, \eta, g)$ is said to be Sasakian manifold if the following holds [6]:

$$(\bar{\nabla}_{E_1}\varphi)E_2 = g(E_1, E_2)\chi - \eta(E_2)E_1 \tag{2.4}$$

$$\bar{\nabla}_{E_1}\chi = -\varphi E_1. \tag{2.5}$$

where $\bar{\nabla}$ is a Riemannian connection.
A Sasakian manifold with constant φ-sectional curvature say c is called Sasakian-space-form. As a generalization of Sasakian-space-form, Alegre et al. [1] introduced the notion of generalized

Sasakian-space-form as that an almost contact metric manifold $\bar{M}^{2d+1}(\varphi,\chi,\eta,g)$ whose curvature tensor $\bar{R}$ of $\bar{M}$ satisfies

$$\begin{aligned}\bar{R}(E_1,E_2)E_3 &= f_1\big\{g(E_2,E_3)E_1-g(E_1,E_3)E_2\big\}+f_2\big\{g(E_1,\varphi E_3)\varphi E_2\\ &- g(E_2,\varphi E_3)\varphi E_1+2g(E_1,\varphi E_2)\varphi E_3\big\}+f_3\big\{\eta(E_1)\eta(E_3)E_2\\ &- \eta(E_2)\eta(E_3)E_1+g(E_1,E_3)\eta(E_2)\chi-g(E_2,E_3)\eta(E_1)\chi\big\}\end{aligned} \tag{2.6}$$

for all $E_1,E_2,E_3\in\Gamma(\bar{M})$ and f_1,f_2,f_3 are certain smooth functions on $\bar{M}$. Such a manifold of dimension $(2d+1)$ satisfying (2.4) and (2.5), is denoted by $\bar{M}^{2d+1}(f_1,f_2,f_3)$. If $f_1=\frac{c+3}{4}$, $f_2=f_3=\frac{c-1}{4}$ then $\bar{M}^{2d+1}(f_1,f_2,f_3)$ reduces to Sasakian-space-form [1].

For an almost contact metric manifold $\bar{M}^{2d+1}(\varphi,\chi,\eta,g)$, we have two naturally defined distribution in the tangent bundle TM of $\bar{M}^{2d+1}(\varphi,\chi,\eta,g)$ as follows [27] $H=ker(\eta)$, $G=span(\chi)$. Then we have $H\oplus G=TM$, $H\cap G=0$ and $H\perp G$. This decomposition allows one to define the Schouten-Van Kampen connection $\tilde{\bar{\nabla}}$ over an almost contact metric structure. The $\tilde{\bar{\nabla}}$ on a generalized Sasakian-space-form $\bar{M}^{2d+1}(f_1,f_2,f_3)$ with respect to $\bar{\nabla}$ is defined by

$$\tilde{\bar{\nabla}}_{E_1}E_2=\bar{\nabla}_{E_1}E_2+\eta(E_2)\varphi E_1-g(\varphi E_1,E_2)\chi \tag{2.7}$$

The generalized Tanaka-Webster connection $\overset{*}{\bar{\nabla}}$ on a generalized Sasakian-space-form $\bar{M}^{2d+1}(f_1,f_2,f_3)$ with respect to $\bar{\nabla}$ is defined by

$$\overset{*}{\bar{\nabla}}_{E_1}E_2=\bar{\nabla}_{E_1}E_2+\eta(E_1)\varphi E_2+\eta(E_2)\varphi E_1-g(\varphi E_1,E_2)\chi. \tag{2.8}$$

The curvature tensor $\tilde{\bar{R}}$ with respect to $\tilde{\bar{\nabla}}$ is given by [16]

$$\begin{aligned}&\tilde{\bar{R}}(E_1,E_2,E_3,E_4)\\ &=f_1\{g(E_2,E_3)g(E_1,E_4)-g(E_2,E_4)g(E_1,E_3)\}\\ &+f_2\{g(E_1,\varphi E_3)g(\varphi E_2,E_4)-g(E_2,\varphi E_3)g(\varphi E_1,E_4)+2g(E_1,\varphi E_2)g(\varphi E_3,E_4)\}\\ &+(f_3+1)\Big[\{\eta(E_1)g(E_2,E_4)-\eta(E_2)g(E_1,E_4)\}\eta(E_3)+\{g(E_1,E_3)\eta(E_2)\\ &-g(E_2,E_3)\eta(E_1)\}\eta(E_4)\Big]+g(E_1,\varphi E_3)g(\varphi E_2,E_4)-g(E_2,\varphi E_3)g(\varphi E_1,E_4).\end{aligned} \tag{2.9}$$

The curvature tensor $\overset{*}{\bar{R}}$ with respect to $\overset{*}{\bar{\nabla}}$ is given by [16]

$$\begin{aligned}&\overset{*}{\bar{R}}(E_1,E_2,E_3,E_4)\\ &=f_1\{g(E_2,E_3)g(E_1,E_4)-g(E_2,E_4)g(E_1,E_3)\}\\ &+(f_2+1)\{g(E_1,\varphi E_3)g(\varphi E_2,E_4)-g(E_2,\varphi E_3)g(\varphi E_1,E_4)\\ &+2g(E_1,\varphi E_2)g(\varphi E_3,E_4)\}+(f_3+1)\Big[\{\eta(E_1)g(E_2,E_4)\\ &-\eta(E_2)g(E_1,E_4)\}\eta(E_3)+\{g(E_1,E_3)\eta(E_2)-g(E_2,E_3)\eta(E_1)\}\eta(E_4)\Big].\end{aligned} \tag{2.10}$$

Let M be an m-dimensional submanifold of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ and ∇, $\nabla^{\perp}$ be the induced connection on $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$ then the Gauss and Weingarten formulas are

$$\bar{\nabla}_{E_1} E_2 = \nabla_{E_1} E_2 + h(E_1, E_2), \tag{2.11}$$

$$\bar{\nabla}_{E_1} V = \nabla^{\perp}_{E_1} V - A_V E_1, \tag{2.12}$$

where $h(E_1, E_2)$, $A_V E_1$ are 2^{nd} fundamental form, shape operator and they are related by [37]

$$g(h(E_1, E_2), V) = g(A_V E_1, E_2). \tag{2.13}$$

From (2.11) and (2.12) we have Gauss and Ricci equation as

$$\begin{aligned} R(E_1, E_2, E_3, E_4) &= \bar{R}(E_1, E_2, E_3, E_4) + g(h(E_1, E_4), h(E_2, E_3)) \\ &- g(h(E_1, E_3), h(E_2, E_4)), \end{aligned} \tag{2.14}$$

$$R^{\perp}(E_1, E_2, V_1, V_2) = \bar{R}(E_1, E_2, V_1, V_2) + g([A_{V_1}, A_{V_2}]E_1, E_2), \tag{2.15}$$

where $E_1, E_2, E_3, E_4 \in \Gamma(TM)$, $V_1, V_2 \in \Gamma(T^{\perp}M)$ and R is the curvature of the submanifold M^d.
A submanifold M^m of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ is said to be invariant submanifold if χ is tangent to M and $\varphi E_1 \in \Gamma(TM)$ for every $E_1 \in TM$ and M^m is said to be anti-invariant if $\chi \in \Gamma(T^{\perp}M)$ and $\varphi E_1 \in \Gamma(T^{\perp}M)$ for every $E_1 \in \Gamma(TM)$. If $m = d$, then anti-invariant submanifold is said to be Legendrian submanifold.
Let $\{b_1, \cdots, b_d\}$ be an orthonormal basis of $\Gamma(T_x M)$ and $\{b_{d+1}, b_{d+2}, \cdots, b_{2d+1} = \chi\}$ be an orthonormal basis of $\Gamma(T_x^{\perp} M)$, then the mean curvature vector Ω is defined by

$$\Omega = \frac{1}{d} \sum_{i=1}^{d} h(b_i, b_i). \tag{2.16}$$

The squared norm of the second fundamental form is defined by

$$\|h\|^2 = \sum_{i,j=1}^{d} g(h(b_i, b_j), h(b_i, b_j)). \tag{2.17}$$

Also we define

$$h_{ij}^r = g(h(b_i, b_j), b_{d+r}). \tag{2.18}$$

From (2.16) we have

$$\|\Omega\|^2 = \frac{1}{d^2} \sum_{r=1}^{d} \left(\sum_{i=1}^{d} h^r(b_i, b_i) \right)^2 = \frac{1}{d^2} \left(\sum_{r=1}^{d} \left(\sum_{i=1}^{d} h_{ii}^r \right)^2 \right). \tag{2.19}$$

By virtue of (2.16), (2.17), (2.18) and (2.19) we have

$$\sum_{r=1}^{d} \sum_{1 \le i < j \le d} h_{ii}^r h_{jj}^r - (h_{ij}^r)^2 = d^2 \|\Omega\|^2 - \|h\|^2. \tag{2.20}$$

We define the normalized scalar curvature for submanifold M^d of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ by

$$\rho = \frac{2\tau}{d(d-1)}, \tag{2.21}$$

where

$$\tau = \sum_{1\leq i\leq j\leq d} R(b_i, b_j, b_j, b_i), \tag{2.22}$$

and normalized normal scalar curvature is

$$\rho^{\perp} = \frac{2}{d(d-1)}\tau^{\perp}, \tag{2.23}$$

where

$$\tau^{\perp} = \sqrt{\sum_{1\leq i\leq j\leq d}\sum_{1\leq \alpha\leq \beta\leq d} (R^{\perp}(b_i, b_j, u_\alpha, u_\beta))^2}, \tag{2.24}$$

and $u_\alpha, u_\beta \in T^{\perp}M$.
The normalized scalar normal curvature is calculated as follows:

$$\rho_N = \frac{2}{d(d-1)}\sqrt{\sum_{1\leq i\leq j\leq d}\sum_{1\leq r\leq s\leq d}\left(\sum_{k=1}^{d}(h^r_{jk}h^s_{ik} - h^r_{ik}h^s_{jk})\right)^2}. \tag{2.25}$$

In similar of (2.21), (2.22), (2.23) and (2.24) we can define $\tilde{\rho}, \tilde{\rho}^{\perp}$ and $\overset{*}{\rho}, \overset{*}{\rho}^{\perp}$ with respect to $\tilde{\nabla}$ and $\overset{*}{\nabla}$ as

$$\tilde{\rho} = \frac{2\tilde{\tau}}{d(d-1)} = \sum_{1\leq i<j\leq m} \frac{2}{d(d-1)}\tilde{R}(b_i, b_j, b_j, b_i), \tag{2.26}$$

$$\tilde{\rho}^{\perp} = \frac{2\tilde{\tau}^{\perp}}{d(d-1)} = \frac{2}{d(d-1)}\sqrt{\sum_{1\leq i<j\leq d}\sum_{1\leq \alpha<\beta\leq n} (\tilde{R}^{\perp}(b_i, b_j, u_\alpha, u_\beta))^2}, \tag{2.27}$$

$$\overset{*}{\rho} = \frac{2\overset{*}{\tau}}{d(d-1)} = \sum_{1\leq i<j\leq m} \frac{2}{d(d-1)}\overset{*}{R}(b_i, b_j, b_j, b_i), \tag{2.28}$$

$$\overset{*}{\rho}^{\perp} = \frac{2\overset{*}{\tau}^{\perp}}{d(d-1)} = \frac{2}{d(d-1)}\sqrt{\sum_{1\leq i<j\leq n}\sum_{1\leq \alpha<\beta\leq n} (\overset{*}{R}^{\perp}(b_i, b_j, u_\alpha, u_\beta))^2}. \tag{2.29}$$

Proposition 2.1. [16] Let M be a C-totally real submanifold of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ with respect to $\tilde{\tilde{\nabla}}$. Then following holds:
(i) $\tilde{h}(E_1, E_2) = h(E_1, E_2)$, $\tilde{\Omega} = \Omega$,
(ii) $\tilde{A}_V E_1 = A_V E_1$,
where $\tilde{h}, \tilde{\Omega}$ and $\tilde{A}$ are second fundamental form, mean curvature and shape operator with respect to $\tilde{\nabla}$.

Proposition 2.2. [16] Let M be a C-totally real submanifold of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ with respect to $\bar{\overset{*}{\nabla}}$. Then following holds:
(i) $\overset{*}{h}(E_1, E_2) = h(E_1, E_2)$, $\overset{*}{\Omega} = \Omega$,
(ii) $\overset{*}{A}_V E_1 = A_V E_1$,
where $\overset{*}{h}, \overset{*}{\Omega}$ and $\overset{*}{A}$ are second fundamental form, mean curvature and shape operator with respect to $\overset{*}{\nabla}$.

Definition 2.3. [3, 4, 12] M of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ is said to be pseudo parallel if (1.2) holds where L_h is a function existing on $U = \{x \in M : \ (h - \omega g)_x \neq 0\}$ for all $E_1, E_2, E_3, E_4 \in \Gamma(TM)$ and

$$\begin{aligned} Q(g,h)(E_3,E_4;E_1,E_2) &= g(E_2,E_4)h(E_1,E_3) - g(E_1,E_4)h(E_2,E_3) \\ &+ g(E_2,E_3)h(E_1,E_4) - g(E_1,E_3)h(E_2,E_4), \end{aligned} \tag{2.30}$$

$$\begin{aligned} (\bar{R}(E_1,E_2)\cdot h)(E_3,E_4) &= R^{\perp}(E_1,E_2)h(E_3,E_4) - h(R(E_1,E_2)E_3,E_4) \\ &- h(E_3,R(E_1,E_2)E_4). \end{aligned} \tag{2.31}$$

If $L_h = 0$, M is considered semi-parallel.

Definition 2.4. [3, 4, 12] M of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ is said to be Ricci generalized pseudo parallel if (1.3) holds where L_S is a function existing on U defined in above and

$$\begin{aligned} Q(S,h)(E_3,E_4;E_1,E_2) &= S(E_2,E_4)h(E_1,E_3) - S(E_1,E_4)h(E_2,E_3) \\ &+ S(E_2,E_3)h(E_1,E_4) - S(E_1,E_3)h(E_2,E_4), \end{aligned} \tag{2.32}$$

where S is the Ricci curvature with respect to ∇ and defined by $S(E_1, E_2) = \sum_{i=1}^{d} R(b_i, E_1, E_2, b_i)$.

Similarly we define

Definition 2.5. A submanifold M of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ with respect to $\tilde{\bar{\nabla}}$ and $\bar{\overset{*}{\nabla}}$ is said to be pseudo parallel if

$$(\tilde{\bar{R}}(E_1,E_2)\cdot h)(E_3,E_4) = L_h Q(g,h)(E_3,E_4;E_1,E_2), \tag{2.33}$$

$$(\bar{\overset{*}{R}}(E_1,E_2)\cdot h)(E_3,E_4) = L_h Q(g,h)(E_3,E_4;E_1,E_2), \tag{2.34}$$

where L_h is a function existing on U and $Q(g,h)(E_3,E_4;E_1,E_2)$ is defined in (2.30) and $(\tilde{\bar{R}}(E_1, E_2)\cdot h)(E_3, E_4)$, $(\bar{\overset{*}{R}}(E_1, E_2)\cdot h)(E_3, E_4)$ are obtained just replacing the term R by $\tilde{R}$ and $\overset{*}{R}$ in (2.31) respectively.
If $L_h = 0$, M is considered semi-parallel.

Definition 2.6. A submanifold M of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ with respect to $\tilde{\bar{\nabla}}$ and $\bar{\overset{*}{\nabla}}$ is said to be Ricci generalized pseudo parallel if

$$(\tilde{\bar{R}}(E_1,E_2)\cdot h)(E_3,E_4) = L_{\tilde{S}} Q(\tilde{S},h)(E_3,E_4;E_1,E_2), \tag{2.35}$$

$$(\overset{*}{\bar{R}}(E_1,E_2)\cdot h)(E_3,E_4)=L_{\overset{*}{S}}Q(\overset{*}{S},h)(E_3,E_4;E_1,E_2), \tag{2.36}$$

where L_S is a function existing on U. $Q(\tilde{S},h)$ and $Q(\overset{*}{S},h)$ are obtained just replacing the term S by $\tilde{S}$ and $\overset{*}{S}$ in (2.32) respectively.

3 Generalized Wintgen ideal Legendrian submanifold of $\bar{M}^{2d+1}(f_1,f_2,f_3)$

Let M^d be a Legendrian submanifold of $\bar{M}^{2d+1}(f_1,f_2,f_3)$, then we have [5]

$$(\rho^{\perp})^2 \quad \le (\|\Omega\|^2-\rho+f_1)^2+\tfrac{2}{d(d-1)}f_2^2+\tfrac{4f_2}{d(d-1)}(\rho-f_1). \tag{3.1}$$

To obtain equality of (3.1) we take orthonormal basis $\{b_1,b_2,\cdots,b_d\}$ of T_xM and orthonormal basis $\{b_{d+1}=\varphi b_1,b_{d+2}=\varphi b_2,\cdots,b_{d+d}=\varphi b_d,b_{2d+1}=\chi\}$ of $T_x^{\perp}M$ on Legendrian submanifolds of $\bar{M}^{2d+1}(f_1,f_2,f_3)$. In Legendrian submanifolds of $\bar{M}^{2d+1}(f_1,f_2,f_3)$ the second fundamental form must satisfy

$$\begin{aligned}
&g(h(b_i,b_j),\varphi b_k)=g(h(b_i,b_k),\varphi b_j)=g(h(b_j,b_k),\varphi b_i),\\
&\text{i.e. } g(h(b_i,b_j),b_{d+k})=g(h(b_i,b_k),b_{d+j})=g(h(b_j,b_k),b_{d+i}),\\
&\text{i.e. } g(A_{b_{d+k}}b_i,b_j)=g(A_{b_{d+j}}b_k,b_i)=g(A_{b_{d+i}}b_j,b_k).
\end{aligned}$$

Now using the above equations for $k=1,i=j=2$; $k=2,i=j=3$; $k=3,i=j=2$ and shape operators of (Lemma 2, [5]), we get the equality of (3.1) holds for some points $x\in M$ if and only if the shape operator takes the following form

$$A_{b_{d+1}}=\begin{pmatrix}0&\mu&0&\cdots&0\\ \mu&0&0&\cdots&0\\ 0&0&0&\cdots&0\\ \vdots&\vdots&\vdots&\cdots&\vdots\\ 0&0&0&\cdots&0\end{pmatrix},\ A_{b_{d+2}}=\begin{pmatrix}\mu&0&0&\cdots&0\\ 0&-\mu&0&\cdots&0\\ 0&0&0&\cdots&0\\ \vdots&\vdots&\vdots&\cdots&\vdots\\ 0&0&0&\cdots&0\end{pmatrix} \tag{3.2}$$

$$A_{b_{d+r}}=O_d, r=3,\cdots,(d+1), \tag{3.3}$$

where μ is real constant on $\mathbb{R}$ [5].

A Legendrian submanifold M^d of $\bar{M}^{2d+1}(f_1,f_2,f_3)$ satisfying the equality of (3.1) is said to be generalized Wintgen ideal Legendrian submanifold [28], we denote such submanifold by $M(\mu)$. Using (2.6), (3.2), and (3.3) we get

$$\begin{cases}
R(b_1,b_2)b_1=-Bb_2,\ R(b_2,b_1)b_1=Bb_2,\\
R(b_1,b_2)b_2=Bb_1,\ R(b_2,b_1)b_2=-Bb_1,\\
R(b_1,b_i)b_i=-R(b_i,b_1)b_i=(B+2\mu^2)b_1\ i=3,\cdots,d,\\
R(b_2,b_i)b_i=-R(b_i,b_2)b_i=(B+2\mu^2)b_2\ i=3,\cdots,d,\\
R(b_i,b_1)b_1=-R(b_1,b_i)b_1=(B+2\mu^2)b_i,\ i=3,\cdots,d,\\
R(b_i,b_2)b_2=-R(b_2,b_i)b_2=(B+2\mu^2)b_i,\ i=3,\cdots,d,\\
R(b_i,b_j)b_i=-R(b_j,b_i)b_i=-(B+2\mu^2)b_j,\ i,j=3,\cdots,d,i\neq j,\\
R(b_i,b_i)b_j=0,\ i,j=1,\cdots,d,
\end{cases} \tag{3.4}$$

where $B = f_1 - 2\mu^2$, R is curvature tensor of M^d.
Similarly using (2.9), (3.2), and (3.3) we get $\tilde{R}(b_i, b_j)b_k$ same as $R(b_i, b_j)b_k$ defined in (3.4).
We now prove the following:

Theorem 3.1. A GWIL submanifold $M(\mu)$ of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ is pseudo parallel if and only if it is totally geodesic.

Proof. Without loss of generality let us put $\{E_1 = b_1, E_2 = b_2, E_3 = b_1, E_4 = b_2\}$ in (1.2) and using (2.30), (2.31), (3.4) and (2.15) we get

$$\left\{2(L_h + f_1 - 3\mu^2) - f_2\right\}\mu = 0. \tag{3.5}$$

Similarly if we put $\{E_1 = b_1, E_2 = b_3, E_3 = b_3, E_4 = b_1\}$ and $\{E_1 = b_1, E_2 = b_3, E_3 = b_1, E_4 = b_2\}$ in (1.2) and using (2.30), (2.31), (3.4) and (2.15) we get the following two equations

$$(L_h + f_1)\mu = 0, \tag{3.6}$$

$$\mu f_2 = 0. \tag{3.7}$$

The above equations (3.5)-(3.7) are consistent if $\mu = 0$ i.e. $M(\mu)$ is totally geodesic. If we take $\mu \neq 0$, then from (3.7) we get $f_2 = 0$. Put $f_2 = 0$ in (3.5), we get $2(L_h + f_1 - 3\mu^2) = 0$. From this and (3.6) we get $\mu = 0$, which is contradiction. Therefore only for $\mu = 0$, then equations (3.5)-(3.7) are consistent. The converse is trivial. Q.E.D.

Corollary 3.2. A GWIL submanifold $M(\mu)$ of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ is semi-parallel if and only if it is totally geodesic.

Proof. If $M(\mu)$ is semi-parallel, then $L_h = 0$. After substitute $L_h = 0$ in (3.5)-(3.7) we get three new equations. The obtained equations are consistent if $\mu = 0$, i.e. $M(\mu)$ is totally geodesic. Q.E.D.

From Proof of Theorem 3.1 we have

Remark 3.3. Let $M(\mu)$ be a GWIL submanifold of $\bar{M}^{2d+1}(f_1, f_2, f_3)$. If $\mu \neq 0$, i.e. $M(\mu)$ is not totally geodesic then $M(\mu)$ is not pseudo parallel.

Remark 3.4. Let $M(\mu)$ be a GWIL submanifold of $\bar{M}^{2d+1}(f_1, f_2, f_3)$. If $\mu \neq 0$, i.e. $M(\mu)$ is not totally geodesic then $M(\mu)$ is not semi-parallel.

Theorem 3.5. A GWIL submanifold $M(\mu)$ of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ is Ricci generalized pseudo parallel if and only if $M(\mu)$ satisfies any one of the following
(i) totally geodesic,
(ii) not totally geodesic but $f_2 = 0$, $f_1 = 0$ and $L_S = -\frac{3}{2}$.

Proof. Without loss of generality let us put $\{E_1 = b_1, E_2 = b_2, E_3 = b_1, E_4 = b_2\}$ in (1.3) and using (2.31), (2.32), (3.4) and (2.15) we get

$$\mu\{2(f_1 - 3\mu^2) - f_2\} + L_S 2\mu\{(d-1)f_1 - 2\mu^2\} = 0. \tag{3.8}$$

Similarly if we put $\{E_1 = b_1, E_2 = b_3, E_3 = b_3, E_4 = b_1\}$ and $\{E_1 = b_1, E_2 = b_3, E_3 = b_1, E_4 = b_2\}$ in (1.3) and using (2.31), (2.32), (3.4) and (2.15) we get the following two equations

$$\mu f_1\{L_S(d-1) + 1\} = 0, \tag{3.9}$$

$$\mu f_2 = 0. \tag{3.10}$$

If $\mu = 0$ i.e. $M(\mu)$ is totally geodesic, then above equations (3.8)-(3.10) are consistent.
If $\mu \neq 0$, then from (3.10), $f_2 = 0$. Also from (3.9) we get $f_1(L_S(d-1)+1) = 0$ i.e. either $f_1 = 0$ or $\{L_S(d-1)+1\} = 0, d \neq 1$. If $d = 1$ then (3.9) we get $f_1 = 0$. Again if $\{L_S(d-1)+1\} = 0, d \neq 1$ then from (3.8) we get $d = \frac{5}{3}$, which is not possible. Therefore from (3.9) we get $f_1 = 0$. Using $f_2 = f_1 = 0$ in (3.8) we get $L_S = -\frac{3}{2}$.
The converse is trivial. Q.E.D.

4 Generalized Wintgen ideal Legendrian submanifold of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ with respect to $\tilde{\tilde{\nabla}}$

Let M^d be a Legendrian submanifold of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ with respect to $\tilde{\tilde{\nabla}}$, then we have [16]

$$(\tilde{\rho}^{\perp})^2 \quad \leq (\|\Omega\|^2 - \tilde{\rho} + f_1)^2 + \tfrac{2}{d(d-1)}(f_2+1)^2 + \tfrac{4(f_2+1)}{d(d-1)}(\tilde{\rho} - f_1). \tag{4.1}$$

This equality holds for some points $x \in M$ if and only if their exists an orthonormal basis $\{b_1, \cdots, b_d\}$ of T_xM and an orthonormal basis $\{b_{d+1} = \varphi b_1, b_{d+2} = \varphi b_2, \cdots, b_{d+d} = \varphi b_d, b_{2d+1} = \chi\}$ of $T_x^{\perp}M$ such that shape operator takes the form (3.2) and (3.3).

Theorem 4.1. A GWIL submanifold $M(\mu)$ of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ with respect to $\tilde{\tilde{\nabla}}$ is pseudo parallel if and only if it is totally geodesic.

Proof. Without loss of generality let us put $\{E_1 = b_1, E_2 = b_2, E_3 = b_1, E_4 = b_2\}$ in (2.33) and using (3.4) we get

$$\left\{2(L_h + f_1 - 3\mu^2) - (f_2+1)\right\}\mu = 0. \tag{4.2}$$

Similarly if we put $\{E_1 = b_1, E_2 = b_3, E_3 = b_3, E_4 = b_1\}$ and $\{E_1 = b_1, E_2 = b_3, E_3 = b_1, E_4 = b_2\}$ in (2.33) and using (3.4) we get the following two equations

$$(L_h + f_1)\mu = 0, \tag{4.3}$$

$$\mu(f_2+1) = 0. \tag{4.4}$$

The above equations (4.2)-(4.4) are consistent if $\mu = 0$, i.e $M(\mu)$ is totally geodesic.
If we take $\mu \neq 0$, then from (4.4) we get $f_2 = -1$. Put $f_2 = -1$ in (4.2), we get $(L_h + f_1 - 3\mu^2) = 0$. From this and (4.3) we get $\mu = 0$, which is contradiction. Therefore only for $\mu = 0$, the equations (4.2)-(4.4) are consistent. The converse is trivial. Q.E.D.

Corollary 4.2. A GWIL submanifold $M(\mu)$ of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ with respect to $\tilde{\tilde{\nabla}}$ is semi-parallel if and only if it is totally geodesic.

Proof. If $M(\mu)$ is semi-parallel, then $L_h = 0$. After substitute $L_h = 0$ in (4.2)-(4.4) we get three new equations. The obtained equations are consistent if $\mu = 0$, i.e. $M(\mu)$ is totally geodesic. Q.E.D.

From the proof of Theorem 4.1, we get

Remark 4.3. Let $M(\mu)$ be a GWIL submanifold of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ with respect to $\tilde{\tilde{\nabla}}$. If $\mu \neq 0$, i.e. $M(\mu)$ is not totally geodesic then $M(\mu)$ is not pseudo parallel.

Remark 4.4. Let $M(\mu)$ be a GWIL submanifold of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ with respect to $\tilde{\bar{\nabla}}$. If $\mu \neq 0$, i.e. $M(\mu)$ is not totally geodesic then $M(\mu)$ is not semi-parallel.

Theorem 4.5. A GWIL submanifold $M(\mu)$ of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ with respect to $\tilde{\bar{\nabla}}$ is Ricci generalized pseudo parallel if and only if $M(\mu)$ satisfies any one of the following
(i) totally geodesic,
(ii) not totally geodesic but $f_2 = -1$, $f_1 = 0$ and $L_{\tilde{S}} = -\frac{3}{2}$.

Proof. Without loss of generality let us put $\{E_1 = b_1, E_2 = b_2, E_3 = b_1, E_4 = b_2\}$ in (2.35) and using (3.4) we get

$$\mu\{2(f_1 - 3\mu^2) - f_2 - 1\} + L_{\tilde{S}} 2\mu\{(d-1)f_1 - 2\mu^2\} = 0. \tag{4.5}$$

Similarly if we put $\{E_1 = b_1, E_2 = b_3, E_3 = b_3, E_4 = b_1\}$ and $\{E_1 = b_1, E_2 = b_3, E_3 = b_1, E_4 = b_2\}$ in (2.35) and using (3.4) we get the following two equations

$$\mu f_1\{L_{\tilde{S}}(d-1) + 1\} = 0, \tag{4.6}$$

$$\mu(f_2 + 1) = 0. \tag{4.7}$$

The above equations (4.5)-(4.7) is consistent if $\mu = 0$, i.e. $M(\mu)$ is totally geodesic. If $\mu \neq 0$, then from (4.7), $f_2 = -1$. Also from (4.6) we get $f_1(L_{\tilde{S}}(d-1) + 1) = 0$ i.e. either $f_1 = 0$ or $\{L_{\tilde{S}}(d-1) + 1\} = 0, d \neq 1$. If $d = 1$ then (4.6) we get $f_1 = 0$. Again if $\{L_{\tilde{S}}(d-1) + 1\} = 0, d \neq 1$ then from (4.5) we get $d = \frac{5}{3}$, which is not possible. Therefore from (4.6) we get $f_1 = 0$. Using $f_2 = -1$ and $f_1 = 0$ in (4.5) we get $L_{\tilde{S}} = -\frac{3}{2}$. The converse of the Theorem holds trivially. Q.E.D.

5 Generalized Wintgen ideal Legendrian submanifold of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ with respect to $\overset{*}{\bar{\nabla}}$

Let M^d be a Legendrian submanifold of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ with respect to $\overset{*}{\bar{\nabla}}$, Then we have [16]

$$(\overset{*}{\rho}^{\perp})^2 \leq (\|\Omega\|^2 - \overset{*}{\rho} + f_1)^2 + \tfrac{2}{d(d-1)}(f_2 + 1)^2 + \tfrac{4(f_2+1)}{d(d-1)}(\overset{*}{\rho} - f_1). \tag{5.1}$$

This equality holds for some points $x \in M$ if and only if their exists an orthonormal basis $\{b_1, \cdots, b_d\}$ of T_xM and an orthonormal basis $\{b_{d+1} = \varphi b_1, b_{d+2} = \varphi b_2, \cdots, b_{d+d} = \varphi b_d, b_{2d+1} = \chi\}$ such that shape operator takes the form (3.2) and (3.3).

Theorem 5.1. A GWIL submanifold $M(\mu)$ of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ with respect to $\overset{*}{\bar{\nabla}}$ is pseudo parallel if and only if it is totally geodesic.

Proof. The proof is similar as the proof of Theorem 4.1. Q.E.D.

Corollary 5.2. A GWIL submanifold $M(\mu)$ of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ with respect to $\overset{*}{\bar{\nabla}}$ is semi-parallel if and only if it is totally geodesic.

Proof. The proof is similar as the proof of Corollary 4.2. Q.E.D.

Remark 5.3. Let $M(\mu)$ be a GWIL submanifold of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ with respect to $\bar{\overset{*}{\nabla}}$. If $\mu \neq 0$, i.e. $M(\mu)$ is not totally geodesic then $M(\mu)$ is not pseudo parallel.

Remark 5.4. Let $M(\mu)$ be a GWIL submanifold of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ with respect to $\bar{\overset{*}{\nabla}}$. If $\mu \neq 0$, i.e. $M(\mu)$ is not totally geodesic then $M(\mu)$ is not semi-parallel.

Theorem 5.5. A GWIL submanifold $M(\mu)$ of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ with respect to $\bar{\overset{*}{\nabla}}$ is Ricci generalized pseudo parallel if and only if $M(\mu)$ satisfies any one of the following
(i)totally geodesic,
(ii) not totally geodesic but $f_2 = -1$, $f_1 = 0$ and $L_{\overset{*}{S}} = -\frac{3}{2}$.

Proof. The proof is similar as the proof of Theorem 4.5. Q.E.D.

6 Conclusion

In this paper, we have shown that GWIL submanifolds of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ are pseudo parallel if GWIL submanifolds is totally geodesic and Ricci generalized pseudo parallel if GWIL submanifolds either totally geodesic or not totally geodesic but $f_2 = 0$, $f_1 = 0$ and $L_S = -\frac{3}{2}$. Also we proved that in GWIL submanifolds of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ with respect to Schouten-Van Kampen connection are pseudo parallel if GWIL submanifolds is totally geodesic and Ricci generalized pseudo parallel if GWIL submanifolds either totally geodesic or not totally geodesic but $f_2 = -1$, $f_1 = 0$ and $L_{\tilde{S}} = -\frac{3}{2}$. Apart from these we proved that in GWIL submanifolds of $\bar{M}^{2d+1}(f_1, f_2, f_3)$ with respect to generalized Tanaka-Webster connection are pseudo parallel if GWIL submanifolds is totally geodesic and Ricci generalized pseudo parallel if GWIL submanifolds either totally geodesic or not totally geodesic but $f_2 = -1$, $f_1 = 0$ and $L_{\overset{*}{S}} = -\frac{3}{2}$

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