Fixed point theorems for generalized interpolative non expansive mappings in $CAT_p(0)$ metric spaces

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Abstract

This paper aims to prove fixed point theorems for generalized interpolative non expansive mappings in $CAT_p(0)$ metric spaces. Also provide a constructive example to support the proven results. The results proved here will be illustrated with an application to Hopf Bifurcations in a Delayed-Energy Based Model of Capital Accumulation.

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1 Introduction

The study of non expansive mapping was introduced by Browder [25] proved the fixed point theorem for non expansive nonlinear operators in a Banach space. A mapping Γ on a subset C of a Banach space X is called a non expansive mapping if

$$\|\Gamma x - \Gamma y\| \quad \leq \quad \|x - y\|, \ \forall \ x, y \in C. \tag{1.1}$$

A non expansive mapping is continuous in its domain.

Kirk [6] proved a fixed point theorem for mappings which do not increases distances. Goebel and Kirk [4] proved an iteration processes for nonexpansive mappings. Kohlenbach and Leustean [7] they proved the results using Mann iterates of directionally nonexpansive mappings in hyperbolic spaces. Suzuki [32] proved the fixed point theorems and convergence theorems for some generalized non-expansive mapping.

In 2007 Goebel and Pineda [18] introduced and studied a new type of mapping called α-non expansive mapping which is a generalization of non-expansive one. For a given mutiindex $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots \lambda_n)$ satisfies $\lambda_1 \geq 0, i = 1, 2, \dots n$ and $\sum_{i=1}^{n} \lambda_i = 1$. Let K be a non empty closed and convex subset of a Banach space X. A mapping $\Gamma : K \to K$ is said to be λ- non expansive if

$$\sum_{i=1}^{n} \lambda_i \|\Gamma^i \kappa - \Gamma^i \nu\| \quad \leq \quad \|\kappa - \nu\|, \forall \kappa, \nu \in K. \tag{1.2}$$

For $\lambda_1 > 0$, the mapping Γ satisfies Lipschitz conditions

$$\|\Gamma \kappa - \Gamma \nu\| \quad \leq \quad \frac{1}{\lambda_1} \|\kappa - \nu\|, \forall \kappa, \nu \in K. \tag{1.3}$$

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Also, inequality (1.2) implies that for $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots \lambda_n)$ the mapping

$$\Gamma_\lambda \kappa = \sum_{i=1}^{n} \lambda_i \Gamma^i \kappa, \forall \kappa \in K, \tag{1.4}$$

is non expansive which is weaker than (1.2). Further, Gromov [1] introduced the concept of CAT(0) spaces. Kirk [11] introduced the fixed point theory in CAT(0) spaces. Goebel and Reich [5] have proved the results on uniform convexity, hyperbolic geometry, and nonexpansive mappings. Reich and Shafrir [3] proved the results of nonexpansive iterations in hyperbolic space. Kirk [11] studied the fixed point theorems in CAT(0) spaces and R-trees. Dhompongsa and Panyanak [2] gave the results on $\triangle$-convergence theorems in CAT (0) spaces. Khamsi and Shukri [12] they gave a generalized CAT(0) spaces. Shukri [9] gave a results on monotone nonexpansive mappings in $CAT_p(0)$ spaces. Darweesh and Shukri [13] proved the fixed points of Suzuki-generalized non-expansive mappings in $CAT_p(0)$ metric spaces.

The novelty of this manuscript is to give the fixed point results for interpolative Ćirić-Rus-Reich-type non expansive mappings in $CAT_p(0)$ metric space with an application to Hopf Bifurcations in a Delayed-Energy Based Model of Capital Accumulation. Our results will extend and generalize several works in literature such as: [13], [6] and [8] and several others.

2 Preliminaries

This section gives definitions, lemmas and some preliminary results which will help to develop the main results.

Kirk [11] and Gromov [1] gave some properties of CAT(0) and CAT(k) spaces as follows: A metric space is called a CAT(0) space if it is geodesically connected and if every triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane.

Definition 2.1. [1, 11] The model spaces M_k^n are defined as:

(i) if $k = 0$, then M_n^0 is the Euclidean space $\mathbb{R}^n$;

(ii) if $k > 0$, then M_n^k is obtained from the sphere $\mathbb{S}^n$ by multiplying the spherical distance by $\frac{1}{\sqrt{k}}$;

(iii) if $k < 0$, then M_n^k is obtained from the hyperbolic $\mathbb{H}^n$ by multiplying the hyperbolic distance by $\frac{1}{\sqrt{-k}}$.

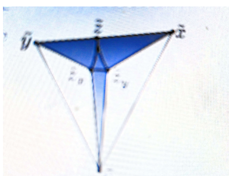

Figure 1: Geodesic Triangle

Khamis and Shukri [12] extended CAT(0) space due to Gromov [1] to $CAT_p(0)$ spaces by considering the comparison triangle belong to a general Banach spaces. In particular the case when the Banach space is l_p, $p \geq 2$.

By defining a geodesic triangle $\triangle(\kappa_1, \kappa_2, \kappa_3)$ in (X, d) in a geodesic metric space (X, d) consists of three points $\kappa_1, \kappa_2, \kappa_3$ in X (the vertices of $\triangle$) and the vertices segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for geodesics triangle $\triangle(\kappa_1, \kappa_2, \kappa_3)$ in (X, d) is a triangle $\bar{\triangle}(\kappa_1, \kappa_2, \kappa_3)' = \triangle(\bar{\kappa_1}, \bar{\kappa_2}, \bar{\kappa_3})$ in the Banach space l_p for $p \geq 2$, such that

$$\|\bar{\kappa_i} - \bar{\kappa_j}\| \quad = \quad d(\kappa_i, \kappa_j), \forall\ i, j \in 1, 2, 3.$$

A point $\bar{\kappa} \in [\bar{\kappa_1}, \bar{\kappa_2}]$ is called a comparison point for $\kappa \in [\kappa_1, \kappa_2]$ if

$$d(\kappa_1, \kappa) \quad = \quad \|\bar{\kappa_1} - \bar{\kappa}\|.$$

Definition 2.2. [12] Let (X, d) be a geodesic metric space. X is said to be a $CAT_P(0)$ space if, for any geodesic triangle $\triangle$ in X, there exists a comparison triangle $\bar{\triangle}$ in l_p, such that the comparison axioms is satisfied i.e., for all $\kappa, \nu \in \triangle$ and comparison points $\bar{\kappa}, \bar{\nu} \in \bar{\triangle}$, we have

$$d(\kappa, \nu) \quad \leq \quad \|\bar{\kappa} - \bar{\nu}\|.$$

An example of $CAT_p(0)$ space from Khamis and Shukra [12].

Example 2.1. Let κ, ν_1, ν_2 be in X and $\frac{\nu_1 \oplus \nu_2}{2}$ is the midpoint of geodesic $[\nu_1, \nu_2]$, then for $p \geq 2$, the comparison axioms implies that

$$d^p(\kappa, \frac{\nu_1 \oplus \nu_2}{2}) \quad \leq \quad \frac{1}{2} d^p(\kappa, \nu_1) + \frac{1}{2} d^p(\kappa, \nu_2) - \frac{1}{2^p} d^p(\nu_1, \nu_2) \tag{2.1}$$

This inequality is known as a (CN_p) inequality of Khamis and Shukra [12].

AS for l_p, for $p > 2$, the (CN_p) inequality implies that $(CN_p)(0)$ metric spaces are uniformly convex with

$$\delta(r, \varepsilon) \quad \geq \quad 1 - (1 - \frac{\varepsilon^p}{2^p})^{\frac{1}{p}}$$

for every $r > 0$ and for each $\varepsilon > 0$. When $p = 2$, the (CN_p) inequality reduces to the classical (CN) inequality of Bruhat and Tits [15]. If z, x, y are points in CAT(0) space and if m is the midpoint of the segment $[\nu_1, \nu_2]$, then the CAT(0) inequality implies

$$d(\kappa, m)^2 \quad \leq \quad \frac{1}{2} d(\kappa, \nu_1)^2 + \frac{1}{2} d(\kappa, \nu_2)^2 - \frac{1}{4} d(\nu_1, \nu_2)^2 \tag{2.2}$$

Definition 2.3. [14] A Banach space X is said to be uniformly convex if for every ε, $0 < \varepsilon \leq 2$, the inequalities

$$\|\kappa\| \leq 1,$$
$$\|\nu\| \leq 1,$$
$$\|\kappa - \nu\| \geq \varepsilon,$$

imply there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\|\frac{\kappa + \nu}{2}\| \leq 1 - \delta.$$

This says that κ and ν are in the closed ball $B_X := \{\kappa \in X : \|\kappa\| \leq 1\}$ with $\|\kappa - \nu\| \geq \varepsilon > 0$, the midpoint of κ and ν lies inside the unit ball B_X at a distance of at least δ from the unit sphere δ_X.

Example 2.2. [14] Every Hilbert space H is a uniformly convex space. Infarct, the parallelogram law gives us

$$\|\kappa + \nu\|^2 \quad = \quad 2(\|\kappa\|^2 + \|\nu\|^2) - \|\kappa - \nu\|^2,$$

for all $\kappa, \nu \in H$.

Suppose $\kappa, \nu \in B_H$ with $\kappa \neq \nu$ and $\|\kappa - \nu\| \geq \varepsilon$. Then

$$\|\kappa - \nu\|^2 \quad \leq \quad 4 - \varepsilon^2$$

So it follows that

$$\|\frac{\kappa + \nu}{2}\| \leq 1 - \delta(\varepsilon),$$

where $\delta(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}$. Therefore H is uniformly convex.

The concepts of $CAT_p(0)$ metric spaces is defined as a metric space (X, d) and a real line $\mathbb{R}$ which define a mapping $\gamma : \mathbb{R} \to X$ in a metric embedding of $\mathbb{R}$ into X if

$$d(\gamma(s), \gamma(t)) = |s - t|, \forall s, t \in \mathbb{R}.$$

The image $\mathbb{R}$ under a metric embedding will be called a metric line. T The image $\gamma([a, b]) \subset X$ will be called a metric segment.

Definition 2.4. [10, 4] Let X be a metric space. A geodesic path in X is a path

$$\gamma : [a, b] \to X.$$

Let $\kappa, \nu \in X$. A metric segment $\gamma([a, b])$ is said to join κ and ν if $\gamma(a) = \kappa$ and $\gamma(b) = \nu$. This shows that (X, d) is of hyperbolic type.

Lemma 2.3. [7] Let $\gamma : \mathbb{R} \to X$ be a metric embedding $a \leq b \in \mathbb{R}$ and $t \in [0, 1]$. Then

(i) $d(\gamma(a), \gamma(\lambda_2 a + \lambda_1 b)) = \lambda_1 d(\gamma(a), \gamma(b))$.

(ii) $d(\gamma(b), \gamma(\lambda_2 a + \lambda_1 b)) = \lambda_2 d(\gamma(a), \gamma(b))$.

Khamis and Shukra [12] introduced some property of a CA_p metric space. In cat(0) metric space (X,d) is said to be convex whenever $[\kappa,\nu]\in K$, for any $\kappa,\nu\in K$. Consider a map $\gamma : X \to \mathbb{R}$ is a type if there exists a bounded sequence $\{\kappa_n\}$ in X such that

$$\gamma(\kappa) \quad = \quad \limsup_{i\to\infty} d(\kappa,\kappa_i).$$

Theorem 2.4. [12] Let (X,d) be a complete $CAT_p(0)$ metric space, with $p \geq 2$. Let K be any non-empty, closed, convex and bounded subset of X. Let γ be a type defined on K. Then any minimizing sequence of γ is convergent. Its limit z is the unique minimum of γ and satisfies,

$$\gamma^p(z) + \frac{1}{2^{p-1}} d^p(z,\kappa) \quad \leq \quad \gamma^p(\kappa), \tag{2.3}$$

for any $\kappa \in K$.

Definition 2.5. [10] Let $\mathcal{E}$ be a vector space. A subset $X \subset \mathcal{E}$ is said to be affinely convex if for all $\kappa,\nu \in X$, the affine segment

$$[\kappa,\nu] := \Big\{\lambda_2\kappa + \lambda_1\nu : \lambda \in [0,1]\Big\}$$

is contained in X.

If X contains a family of metric segments, such that for each pair of distinct points κ and ν in X, there is a unique metric line which joins κ and ν. Denoted by $[\kappa,\nu]$ or $[\nu,\kappa]$ the unique metric segment joining the two points κ and ν from X. This shows that $[\kappa,\kappa] = \{\kappa\}$, for any $\kappa \in X$.

Proposition 2.5. [7] Let (X,d) be a metric space of hyperbolic type. Let $\kappa,\nu \in X$. For each $\lambda \in [0,1]$, there is a unique point $z \in [\kappa,\nu]$, such that

$$d(\kappa,z) \quad = \lambda_1 d(\kappa,\nu),$$

and

$$d(\nu,z) \quad = \lambda_2 d(\kappa,\nu),$$

such points will be denoted by

$$z \quad = \quad \lambda_2\kappa \oplus \lambda_1\nu.$$

For $z \in [\kappa,\nu]$ this imply that

$$d(\kappa,z) + d(z,\nu) \quad = \quad d(\kappa,\nu).$$

Definition 2.6. [5] (X,d) is a hyperbolic metric space if

(i) $d(\lambda_2\kappa \oplus \lambda_1\nu, (\lambda_2\kappa \oplus \lambda_1 z) \leq \lambda_1 d(\nu,z),$

(ii) $d(\lambda_2\kappa \oplus \lambda_1\nu, \lambda_2 z \oplus \lambda w) \leq \lambda_2 d(\kappa, z) + \lambda_1 d(\nu, w)$,

for any $\lambda_1, \lambda_2 \in [0,1]$ and all $\kappa, \nu, z, w \in X$. Note that every hyperbolic space is a space of hyperbolic type [7].

Examples of normed spaces which are hyperbolic metric space.

(i) Hadamard manifolds [5].

(ii) The Hilbert open unit ball [6].

(iii) $CAT_p(0)$ metric space [12].

Opial's [24] introduced an inequality for well convergent sequence characterizing its limits:

Definition 2.7. Let X be a Banach spaces. X satisfies Opial's condition if for each κ in X and each sequence $\{\kappa_n\}$ weakly convergent to κ.

$$\liminf_{n\to\infty} \|\kappa_n - \nu\| \quad > \quad \liminf_{n\to\infty} \|\kappa_n - \kappa\|, \tag{2.4}$$

holds for $\nu \neq \kappa$.

Browder [25] obtained an equivalent definition by replacing (2.4) by

$$\limsup_{n\to\infty} \|\kappa_n - \nu\| \quad > \quad \limsup_{n\to\infty} \|\kappa_n - \kappa\|. \tag{2.5}$$

Naor and Silberman [17] extended p-uniform convexity to the set of geodesic space in the following way:

Definition 2.8. Fix $1 < p < \infty$. A metric space (X, d) is called p-uniformly convex with parameter $C > 0$ iff (X, d) is geodesic and for any three points $\kappa, \nu, z \in X$ with $c = 2$ and $\lambda_i \in [0,1]$, for all $i = 1, 2$.

$$d^p(\lambda_2\kappa \oplus \lambda_1\nu, z) \quad \leq \quad \lambda_2 d^p(\kappa, z) + \lambda_1 d^p(\nu, z) - \frac{c}{2}\lambda_1\lambda_2 d^p(\kappa, \nu). \tag{2.6}$$

Note that the inequality above guarantees that the space X is uniquely geodesic.

Following the definition of uniquely point $(1-\lambda)\kappa \oplus \lambda\nu$ on a geodesic segment $[\kappa, \nu]$ and take $\lambda_1 + \lambda_2 = 1$, we have the following notation

$$\oplus_{i=1}^{2}\lambda_i\kappa_i \quad = \quad \frac{\lambda_1}{\lambda_1+\lambda_2}\kappa_1 \oplus \frac{\lambda_2}{\lambda_1+\lambda_2}\kappa_2,$$

through induction we can write

$$\oplus_{i=1}^{n}\lambda_i\kappa_i \quad = \quad (1-\lambda_n) \oplus \lambda_n\kappa_n\Big[\frac{\lambda_1}{1-\lambda_n}\kappa_1 \oplus \frac{\lambda_2}{1-\lambda_n}\kappa_2 \oplus \cdots \oplus \frac{\lambda_{n-1}}{1-\lambda_n}\kappa_{n-1}\Big].$$

Calderon *et al.* [16] gave the following lemma.

Lemma 2.6. Let X be a $CAT_p(0)$ space, with $p \geq 2$ with $\kappa, \kappa_i \in X$ and $\lambda_i \in [0,1]$ for $i = 1, 2, \ldots n (n \geq 2)$ such that $\sum_{i=1}^{n} \lambda_i = 1$. Then

(i) $d(\oplus_{i=1}^{n} \lambda_i \kappa_i, \kappa) \leq \sum_{i=1}^{n} \lambda_i d(\kappa_i, \kappa)$,

(ii) $d^p(\oplus_{i=1}^{n} \lambda_i \kappa_i, \kappa) \leq \sum_{i=1}^{n} \lambda_i d^p(\kappa_i, \kappa) - \frac{1}{2^{p-1}} \lambda_i \lambda_j d^p(\kappa_i, \kappa_j)$.

Therefore Lemma 2.6, yields to the following inequality;

$$d(\lambda_1\lambda_1 \oplus \lambda_2\lambda_2 \oplus \cdots \oplus \lambda_n\kappa_n, \lambda_1\nu_1 \oplus \lambda_2\nu_2 \oplus \cdots \oplus \nu_n\lambda_n) \leq \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i\lambda_j d(\kappa_i, \nu_j),$$

$$d(\oplus_{i=1}^{n} \lambda_i\kappa_i, \oplus_{j=1}^{n} \lambda_j\nu_j) \leq \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i\lambda_j d(\kappa_i, \nu_j),$$

$\forall \kappa_i \ldots \kappa_n$ *and* $\nu_i \ldots \nu_n \in X$.

Further, Goebel and Pineda [18] introduced the class of (λ, p)-nonexpansive mappings in Banach spaces [18] as follows.

Definition 2.9. [18] A function $\Gamma : K \to K$ is called (λ, p)-non expansive mapping if for some $\lambda = (\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n)$ with $\sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0$ for all $1 \leq i \leq n, \lambda_1, \lambda_n > 0$, and for some $p \in [1, \infty)$,

$$\sum_{i=1}^{n} \lambda_i d^p(\Gamma^i\kappa, \Gamma^i\nu) \leq d^p(\kappa, \nu) \ \forall \ \kappa, \nu \in K. \tag{2.7}$$

In particular, for $n = 2$ using inequality (2.7),we have

$$\lambda_1 d^p(\Gamma\kappa, \Gamma\nu) + \lambda_2 d^p(\Gamma^2\kappa, \Gamma^2\nu) \leq d^p(\kappa, \nu) \ \forall \ \kappa, \in K, \tag{2.8}$$

we say that Γ is $((\lambda_1, \lambda_2), p)$-nonexpansive mapping.

Equivalent to

$$\Gamma_\lambda\kappa = \sum_{i=1}^{2} \lambda_i\Gamma^i\kappa = \oplus_{i=1}^{2} \lambda_i\Gamma^i\kappa = \lambda_1\Gamma\kappa + \lambda_2\Gamma^2\kappa \tag{2.9}$$

$\forall \kappa \in K$.

The following example introduced by Khan *et al.* [20], which satisfies inequality (2.8).

Example 2.7. Let $X = \{0, 1, 2\}$ with usual metric $d(\kappa, \nu) = |\kappa - \nu|$ for alll $\kappa, \nu \in X$. Define the mapping $\Gamma : X \to X$ by

$$\Gamma\kappa = \begin{cases} 1, & \kappa \neq 0, \\ 0, & \kappa = 0. \end{cases} \tag{2.10}$$

Setting $\lambda = (\lambda_1, \lambda_2), \lambda_1, \lambda_2 > 0$ and $\lambda_1 + \lambda_2 = 1$, for $p \geq 1$, we have

$$\lambda_1 |\Gamma\kappa - \Gamma\nu|^p + \lambda_2 |\Gamma^2\kappa - \Gamma^2\nu|^p \quad \leq \quad |\kappa - \nu|^p \ \forall \ \kappa, \in K, \tag{2.11}$$

Therefore Γ is $((\lambda_1, \lambda_2), p)$-nonexpansive mapping.

Similarly, for $n = 3$ using inequality (2.7),we have

$$\lambda_1 d^p(\Gamma\kappa, \Gamma\nu) + \lambda_2 d^p(\Gamma^2\kappa, \Gamma^2\nu) + \lambda_3 d^p(\Gamma^3\kappa, \Gamma^3\nu) \quad \leq \quad d^p(\kappa, \nu) \ \forall \ \kappa, \in K, \tag{2.12}$$

we say that Γ is $((\lambda_1, \lambda_2, \lambda_3), p)$-nonexpansive mapping.
Leads to

$$\Gamma_\lambda \kappa \quad = \quad \sum_{i=1}^{3} \lambda_i \Gamma^i \kappa = \oplus_{i=1}^{3} \lambda_i \Gamma^i \kappa = \lambda_1 \Gamma\kappa + \lambda_2 \Gamma^2\kappa + \lambda_3 \Gamma^3\kappa, \tag{2.13}$$

$\forall \kappa \in K$.

For (λ, p)-nonexpansive mapping with respect to the metric d, for all $\kappa, \nu \in K$, we have

$$d^p(\kappa, \nu) \quad \leq \quad \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} \lambda_i \Big) d^p(\Gamma^{j-1}\kappa, \Gamma^{j-1}\nu).$$

When $n = 1, i = 1$ and $j = 1$, Γ is called $(\lambda, 1)$-non expansive, that is

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i d^p(\Gamma^i\kappa, \Gamma^i\nu)^p &\leq d^p(\kappa, \nu), \\ d^p(\Gamma\kappa, \Gamma\nu) &\leq \frac{1}{\lambda_1} d^p(\kappa, \nu), \\ d(\Gamma\kappa, \Gamma\nu) &\leq \frac{1}{(\lambda_1)} d(\kappa, \nu), . \end{aligned}$$

By induction, we obtain

$$d(\Gamma^i\kappa, \Gamma^i\nu) \quad \leq \quad \Big(\frac{1}{\lambda_1}\Big)^i d(\kappa, \nu).$$

The following is an example of (λ, p)-nonexpansive mapping

Example 2.8. [19] Let $K = \{(\kappa, \nu) \in \mathbb{R}^2 : \kappa, \nu \geq 0, \kappa^2 + \nu^2 \leq 1\}$ and $\Gamma(\kappa, \nu) = (\nu, \kappa)$ for all $\kappa, \nu \in K$ with Euclidean metric. Then Γ is an $((\lambda_1, \lambda_2), p)$-nonexpansive.

The following lemma introduced by Asadi *et al.* [19] as below.

Lemma 2.9. [19] If Γ is an (λ, p)-metrically invariant mapping, then the mapping Γ_λ is a non expansive mapping, then

$$\begin{aligned} d(\Gamma_\lambda\kappa, \Gamma_\lambda\nu) &= d(\lambda_1\Gamma\kappa \oplus \lambda_2\Gamma^2\kappa, \lambda_1\Gamma\nu \oplus \lambda_2\Gamma^2\nu), \\ d(\Gamma_\lambda\kappa, \Gamma_\lambda\nu) &= \lambda_1 d(\Gamma\kappa, \Gamma\nu) + \lambda_2 d(\Gamma^2\kappa, \Gamma^2\nu), \\ d(\Gamma_\lambda\kappa, \Gamma_\lambda\nu) &\leq d(\kappa, \nu). \end{aligned}$$

Lemma 2.10. [32] Let K be a non empty subset of a metric space X. Suppose $\Gamma : K \to K$ is a Suzuki generalized non expansive mapping. Then

$$d(\kappa, \Gamma\nu) \leq 3d(\Gamma\kappa, \kappa) + d(\kappa, \nu),$$

for all $\kappa, \nu \in K$.

Definition 2.10. [22] A self mapping Γ on X is said to be asymptotically regular at a point $\kappa \in X$ if

$$\lim_{i\to\infty} d(\Gamma^i\kappa, \Gamma^{i+1}\kappa) = 0. \tag{2.14}$$

Lemma 2.11. [21] If X is a metric space and Γ is an asymptotically regular self mapping on X, that is

$$d(\Gamma^i\kappa, \Gamma^{i+1}\kappa) \to 0, \ \forall\ \kappa \in X, \tag{2.15}$$

then Γ has the approximated fixed point property.

Definition 2.11. [23] The self map Γ on X is called a generalized convex contraction whenever there exists a mapping and $\lambda_1, \lambda_2 \in [0, 1]$, with $\lambda_1 + \lambda_2 < 1$, such that

$$d(\Gamma^2\kappa, \Gamma^2\nu) \leq \lambda_1 d(\kappa, \nu) + \lambda_2 d(\Gamma\kappa, \Gamma\nu),$$

for all $\kappa, \nu \in X$.

The following are some preliminaries results.

Browder [25] used inequality (1.1) to prove the following theorem.

Theorem 2.12. [25] If K is a non empty bounded closed convex subset of a uniformly Banach space E and $\Gamma : K \to K$ is a non expansive mapping, the Γ has a fixed point. Moreover, the fixed point set of Γ is a closed and convex subset of K.

Darweesh and Shukri [13] proved the following theorems on $CAT_p(0)$ space.

Theorem 2.13. [13] Let K be a non-empty bounded closed convex subset of a complete $CAT_p(0)$ metric space M with $p \geq 2$. Suppose that $\Gamma : K \to K$ is a Suzuki generalized non expansive mapping, then Γ has a fixed point.

Calderen *et al* [16] extended the above theorem in $CAT_p(0)$-metric spaces as follows

Definition 2.12. [16] Let K be a non-empty bounded closed convex subset of a $CAT_p(0)$-metric spaces. $\Gamma : K \to K$ be an α-non expansive if, for some mutiindex $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots \alpha_n)$ such that $\alpha_1 \geq 0, i = 1, 2, \dots n$ and $\sum_{i=1}^{n} \alpha_i = 1$, we have

$$\oplus_{i=1}^{n} \alpha_i d(\Gamma^i \kappa, \Gamma^i \nu) \leq d(\kappa, \nu), \forall \kappa, \nu \in K. \tag{2.16}$$

Since the first coefficient $\alpha_1 > 0$, the mapping Γ satisfies Lipschitz conditions

$$d(\Gamma \kappa, \Gamma \nu) \leq \frac{1}{\alpha_1} d(\kappa, \nu), \forall \kappa, \nu \in K. \tag{2.17}$$

Trivially, $\Gamma_\alpha : X \to X$ defined by

$$\Gamma_\alpha \kappa = \oplus_{i=1}^{n} \alpha_i \Gamma^i \kappa, \forall \kappa \in K. \tag{2.18}$$

Theorem 2.14. [16] Let K be a non empty closed and convex subset of a $CAT_p(0)$-metric spaces. $\Gamma : K \to K$ be an α-non expansive if, for some mutiindex $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots \alpha_n)$ such that $\alpha_1 > \frac{1}{n - \sqrt[n]{2}}, i = 1, 2, \dots n$ and $\sum_{i=1}^{n} \alpha_i = 1$, then the set of fixed points $\Gamma, F(\Gamma)$, coincides with the set of fixed points $\Gamma_\alpha = F(\Gamma_\alpha)$.

The following definitions and theorems was proved by Khan *et al.* [20].

Definition 2.13. A self mapping Γ on X is said to be (λ_1, p)-contraction, if for some $\lambda_1 \in (0, 1)$ and $p \geq 1$, there exists $0 \leq \beta < 1$ satisfying the following inequality

$$\lambda_1 d^p(\Gamma \kappa, \Gamma \nu) + (1 - \lambda_1) d^p(\Gamma^2 \kappa, \Gamma^2 \nu) \leq \beta d^p(\kappa, \nu) \ \forall \ \kappa, \in K, \tag{2.19}$$

Theorem 2.15. Let (X, d) be a metric space and $\Gamma : X \to X$ be (λ_1, p)-contraction such thst $\beta + \lambda_1 < 1$. Then, Γ has the approximated fixed point property. Further, if (X, d) is a complete metric space, then Γ has a unique fixed point.

Next, they proved the following results.

Definition 2.14. A self mapping Γ on X is said to be (λ_1, p)-contraction, if for some $\lambda_1 \in (0, 1)$ and $p \geq 1$, there exists $\beta_i \geq 0$ for all $\sum_{i=1}^{5} < 1$ satisfying the following inequality

$$\begin{aligned} \lambda_1 d^p(\Gamma \kappa, \Gamma \nu) + (1 - \lambda_1) d^p(\Gamma^2 \kappa, \Gamma^2 \nu) \leq \ & \beta_1 d^p(\kappa, \nu) + \beta_2 d^p(\kappa, \Gamma \kappa) + \beta_3 d^p(\Gamma \kappa, \Gamma^2 \kappa) \\ & + \beta_4 d^p(\nu, \Gamma \nu) + \beta_5 d^p(\nu, \Gamma^2 \nu) \end{aligned}$$

$\forall \ \kappa, \in K.$

Theorem 2.16. Let (X, d) be a metric space and $\Gamma : X \to X$ be (λ_1, p)-convex contraction such that $\sum_{i=1}^{5} \beta_i + \lambda_1 < 1$. Then, Γ has the approximated fixed point property. Further, if (X, d) is a complete metric space, then Γ has a unique fixed point.

The study of interpolative mapping using Kanann contraction wa introduced by Karapinar [26] as fllows:

Definition 2.15. [26] Let (X, d) be a metric space, a mapping $\Gamma : X \to X$ is said to be interpolative Kannan contraction mappings if

$$d(\Gamma x, \Gamma y) \quad \leq \quad \eta[d(x, \Gamma x)]^{\delta}.[d(y, \Gamma y)]^{1-\delta}, \tag{2.20}$$

for all $x, y \in X$ with $x \neq \Gamma x$, where $\eta \in [0, 1)$ and $\delta \in (0, 1)$.

Theorem 2.17. [26] Let (X, d) be a complete metric space and Γ be an interpolative Kannan-type contraction. Then Γ has a unique fixed point in X.

Karapinar *et al.* [27] proved the result for an interpolative Hardy-Rogers type contractions in metric space as follows:

Definition 2.16. [27] Let (X, d) be a metric space. For a self mapping $\Gamma : X \to X$ is called an interpolative Hardy-Rogers type contraction if there exists $c \in [0, 1)$ and $\delta, \mu, \gamma \in (0, 1)$ with $\delta + \mu + \gamma < 1$, such that

$$\begin{aligned} d(\Gamma x, \Gamma y) \quad &\leq \quad c\Big[d(x, y)\Big]^{\delta}.\Big[d(x, Tx)\Big]^{\mu}.\Big[d(y, \Gamma y)\Big]^{\gamma}. \\ &\Big[\frac{1}{2}(d(x, \Gamma y) + d(y, \Gamma x))\Big]^{1-\delta-\mu-\gamma}, \end{aligned} \tag{2.21}$$

for all $x, y \in X \backslash Fix(\Gamma)$.

Using Definition 2.16, Karapinar *et al.* [27] proved the following theorem:

Theorem 2.18. Let (X, d) be a complete metric space and Γ be an interpolative Hardy-Rogers type contraction. Then Γ has a fixed point in X.

Further, Mohammadi *et al.* [28] gave the following definition and theorem for the extended interpolative Ćirić-Reich-Rus type F-contraction mappings in metric space.

Definition 2.17. [28] Let (X, d) be a metric space, we say that the self-mapping $\mathrm{T} : X \to X$ is an extended interpolative Ćirić-Reich-Rus type F-contraction mappings if there exists $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$, $\tau > 0$ and $F \in \mathcal{F}$ such that

$$\tau + F(d(\mathrm{T}u, \mathrm{T}v)) \quad \leq \quad \alpha F(d(u, v)) + \beta F(d(u, \mathrm{T}u)) + (1 - \alpha - \beta)F(d(v, \mathrm{T}v)),$$

for all $u, v \in X \backslash Fix(\mathrm{T})$ with $u \neq \mathrm{T}u$ with $d(\mathrm{T}u, \mathrm{T}v) > 0$.

Theorem 2.19. [28] Let (X, d) be a complete metric space and T be an interpolative Ćirić-Reich-Rus type F-contraction. Then T admits a fixed point in X.

Dass and Gupta [29] proved the following results on metric space:

Theorem 2.20. [29] Let (X, d) be a complete complex metric space and $T : X \to X$ be a mappings such that

(i)

$$d(Tx, Ty) \quad \leq \quad \beta d(x, y) + \alpha \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)},$$

for all $x, y \in X$, where α, β are non-negative real with $\alpha + \beta < 1$ and

(ii) for some $x_0 \in X$, the sequence of iterates $\{T^n x_0\}$ has a subsequences $\{T^{n_k} x_0\}$ with $z = \lim_{n\to\infty} T^{n_k} x_0$.

Then z is a unique fixed point of T.

3 Main results

This section start with the following definition.

Definition 3.1. A self mapping Γ on a $CAT_p(0)$-metric spaces is said to be $((\lambda_1, \lambda_2), p)$-contraction, if for some $\lambda_i \in (0, 1)$ and $p \geq 1$, there exists $0 \leq \beta < 1$ and $\sum_{i=1}^{2} \lambda_i = 1$ satisfying the following inequality

$$\lambda_1 d^p(\Gamma\kappa, \Gamma\nu) + \lambda_2 d^p(\Gamma^2\kappa, \Gamma^2\nu) \quad \leq \quad \beta d^p(\kappa, \nu) \tag{3.1}$$

$\forall\ \kappa, \in K$.

Theorem 3.1. Let K be a non-empty bounded closed weakly uniformly convex subset of a $CAT_p(0)$-metric spaces and $\Gamma : K \to K$ be $((\lambda_1, \lambda_2), p)$-contraction such that $\beta + \lambda_i < 1, i = 1, 2$. Then, Γ has the approximated fixed point property. Further, if $R > 0$ and $p \geq 1$ with Fi x (T) is a closed and convex subset of C. Then Γ has a unique fixed point.

Proof. Let K be a bounded closed convex subset of X, where X is a $CAT_p(0)$-metric spaces. Let $\kappa_0 \in K$. Now, define a sequence $\{\kappa_i\}$ by $\kappa_{i+1} = \Gamma^{i+1}\kappa_0$ for all $i \geq 0$. If $\kappa_i = \kappa_{i+1}$ i.e., $\Gamma^i\kappa_0 = \Gamma^{i+1}\kappa_0$ for some i, then $\Gamma^i\kappa_0$ is a fixed point of Γ. On contrary to that, assume $\Gamma^i\kappa_0 \neq \Gamma^{i+1}\kappa_0$. Let $\kappa = \kappa_0$ and $\nu = \Gamma\kappa_0$ in the inequality (3.1), we obtain

$$\lambda_1 d^p(\Gamma\kappa_0, \Gamma^2\kappa_0) + \lambda_2 d^p(\Gamma^2\kappa_0, \Gamma^3\kappa_0) \quad \leq \quad \beta d^p(\kappa_0, \Gamma\kappa_0) \tag{3.2}$$

By using Definition 2.11 for $\kappa = \kappa_0$ and $\nu = \Gamma\kappa_0$, we obtain

$$d^p(\Gamma^2\kappa_0, \Gamma^3\kappa_0) \quad \leq \quad \lambda_1 d^p(\kappa_0, \Gamma\kappa_0) + \lambda_2 d^p(\Gamma\kappa_0, \Gamma^2\kappa_0). \tag{3.3}$$

Applying (3.3) in (3.2), gives

$$\begin{aligned}
\lambda_1 d^p(\Gamma\kappa_0, \Gamma^2\kappa_0) + \lambda_2(\lambda_1 d^p(\kappa_0, \Gamma\kappa_0) + \lambda_2 d^p(\Gamma\kappa_0, \Gamma^2\kappa_0)) &\leq \beta d^p(\kappa_0, \Gamma\kappa_0) \\
\lambda_1 d^p(\Gamma\kappa_0, \Gamma^2\kappa_0) + \lambda_2\lambda_1 d^p(\kappa_0, \Gamma\kappa_0) + \lambda_2^2 d^p(\Gamma\kappa_0, \Gamma^2\kappa_0) &\leq \beta d^p(\kappa_0, \Gamma\kappa_0) \\
(\lambda_1 + \lambda_2^2) d^p(\Gamma\kappa_0, \Gamma^2\kappa_0) + \lambda_1\lambda_2 d^p(\kappa_0, \Gamma\kappa_0) &\leq \beta d^p(\kappa_0, \Gamma\kappa_0)
\end{aligned}$$

Equivalently to

$$\begin{aligned}(\lambda_1+\lambda_2^2)d^p(\Gamma\kappa_0,\Gamma^2\kappa_0) &\leq (\beta-\lambda_1\lambda_2)d^p(\kappa_0,\Gamma\kappa_0)\\ d^p(\Gamma\kappa_0,\Gamma^2\kappa_0) &\leq \frac{(\beta-\lambda_1\lambda_2)}{(\lambda_1+\lambda_2^2)}d^p(\kappa_0,\Gamma\kappa_0).\end{aligned} \tag{3.4}$$

Putting $\vartheta=\frac{(\beta-\lambda_1\lambda_2)}{(\lambda_1+\lambda_2^2)}$ in (3.4), we get

$$d^p(\Gamma\kappa_0,\Gamma^2\kappa_0) \leq \vartheta d^p(\kappa_0,\Gamma\kappa_0). \tag{3.5}$$

Similarly, let $\kappa=\Gamma\kappa_0$ and $\nu=\Gamma^2\kappa_0$ in the inequality (3.1), we get

$$\lambda_1 d^p(\Gamma^2\kappa_0,\Gamma^3\kappa_0)+\lambda_2 d^p(\Gamma^3\kappa_0,\Gamma^4\kappa_0) \leq \beta d^p(\Gamma\kappa_0,\Gamma^2\kappa_0) \tag{3.6}$$

and

$$d^p(\Gamma^3\kappa_0,\Gamma^4\kappa_0) \leq \lambda_1 d^p(\Gamma\kappa_0,\Gamma^2\kappa_0)+\lambda_2 d^p(\Gamma^2\kappa_0,\Gamma^3\kappa_0) \tag{3.7}$$

By using (3.7) in (3.6), we get

$$\begin{aligned}\lambda_1 d^p(\Gamma^2\kappa_0,\Gamma^3\kappa_0)+\lambda_2(\lambda_1 d^p(\Gamma\kappa_0,\Gamma^2\kappa_0)+\lambda_2 d^p(\Gamma^2\kappa_0,\Gamma^3\kappa_0)) &\leq \beta d^p(\Gamma\kappa_0,\Gamma^2\kappa_0)\\ \lambda_1 d^p(\Gamma^2\kappa_0,\Gamma^3\kappa_0)+\lambda_1\lambda_2 d^p(\Gamma\kappa_0,\Gamma^2\kappa_0)+\lambda_2^2 d^p(\Gamma^2\kappa_0,\Gamma^3\kappa_0) &\leq \beta d^p(\Gamma\kappa_0,\Gamma^2\kappa_0)\\ (\lambda_1+\lambda_2^2)d^p(\Gamma^2\kappa_0,\Gamma^3\kappa_0)+\lambda_1\lambda_2 d^p(\Gamma\kappa_0,\Gamma^2\kappa_0) &\leq \beta d^p(\Gamma\kappa_0,\Gamma^2\kappa_0)\end{aligned}$$

Leads to

$$\begin{aligned}(\lambda_1+\lambda_2^2)d^p(\Gamma^2\kappa_0,\Gamma^3\kappa_0) &\leq (\beta-\lambda_1\lambda_2)d^p(\Gamma\kappa_0,\Gamma^2\kappa_0)\\ d^p(\Gamma^2\kappa_0,\Gamma^3\kappa_0) &\leq \frac{(\beta-\lambda_1\lambda_2)}{(\lambda_1+\lambda_2^2)}d^p(\Gamma\kappa_0,\Gamma^2\kappa_0).\\ d^p(\Gamma^2\kappa_0,\Gamma^3\kappa_0) &\leq \vartheta d^p(\Gamma\kappa_0,\Gamma^2\kappa_0).\end{aligned} \tag{3.8}$$

Taking (3.5) in (3.8), we obtain

$$d^p(\Gamma^2\kappa_0,\Gamma^3\kappa_0) \leq \vartheta^2 d^p(\kappa_0,\Gamma\kappa_0).$$

By induction we have

$$d^p(\Gamma^2\kappa_0,\Gamma^3\kappa_0) \leq \vartheta^i d^p(\kappa_0,\Gamma\kappa_0).$$

From Definition 2.9, we obtain

$$d^p(\Gamma^i\kappa_0,\Gamma^{i+1}\kappa_0) \leq \vartheta^i d^p(\kappa_0,\Gamma\kappa_0).$$

Therefore, $d^p(\Gamma^i\kappa_0,\Gamma^{i+1}\kappa_0)\to 0$ as $i\to\infty$, i.e., Γ is asymptotically regular at κ_0. By Lemma 2.11 and Definition 2.10, Γ has approximated fixed point property.

Suppose that Γ is closed, bounded and uniformly convex and (X,d) is a complete $CAT_p(0)$-metric space.

Now, to show that $\{\kappa_i\}$ is a Cauchy sequence. Let $\{\kappa_i\}$ be a sequence in K such that$\gamma(\kappa) = \lim_{i\to\infty} d(\kappa_j,\kappa)$. Denote $\gamma_0 = \inf\{\gamma(\kappa) : \kappa \in K\}$. Let $\{\nu_i\}$ be a minimized sequence of γ. Since K is bounded, there exists $R > 0$ such that $d(\kappa,\nu) \leq R$ for any $\kappa,\nu \in K$. Since (X,d) is a $CAT_p(0)$-metric space, Definition 2.8 implies

$$d^p(\kappa_i, \lambda_2\nu_i \oplus \lambda_1\nu_j) \quad \leq \quad \lambda_2 d^p(\kappa_i,\nu_i) + \lambda_1 d^p(\nu_j,\kappa_i) - \lambda_1\lambda_2 d^p(\nu_i,\nu_j). \tag{3.9}$$

Using inequality 2.3, we have

$$\gamma^p(\kappa_i, \lambda_2\nu_i \oplus \lambda_1\nu_j) \quad \leq \quad \lambda_2\gamma^p\nu_i + \lambda_1\gamma^p\nu_j - \lambda_1\lambda_2\gamma^p(\nu_i,\nu_j),$$

which implies

$$\gamma_0^p \quad \leq \quad \lambda_2\gamma^p\nu_i + \lambda_1\gamma^p\nu_j - \lambda_1\lambda_2\gamma^p(\nu_i,\nu_j),$$

for all $\nu_{i,j} \geq 1$. Since $\{\nu_i\}$ is the minimizing sequence of γ, we conclude that

$$\lim_{i,j\to\infty} d(\nu_i,\nu_j) \quad = \quad 0.$$

Hence the sequence $\{\nu_i\}$ is a Cauchy sequence.

Since γ is continuous and X is complete, $\{\kappa_n\}$ converges to some $z \in K$ such that $\gamma_0 = \gamma(z)$. By Lemma 2.10, we have

$$\begin{aligned} d^p(z,\Gamma\kappa_n) \quad &\leq \quad 3d^p(\Gamma\kappa_n,\kappa_n) + d^p(z,\kappa_n), \\ \limsup_{i\to\infty} d^p(z,\Gamma\kappa_n) \quad &\leq \quad 3\limsup_{i\to\infty} d^p(\Gamma\kappa_n,\kappa_n) + \limsup_{i\to\infty} d^p(z,\kappa_n). \end{aligned}$$

Since $\{\kappa_n\}$ is an approximated fixed point of Γ, we get

$$\begin{aligned} \limsup_{i\to\infty} d^p(z,\Gamma\kappa_n) \quad &\leq \quad \limsup_{i\to\infty} d^p(z,\kappa_n), \\ \limsup_{i\to\infty} d^p(z,\Gamma z) \quad &\leq \quad \limsup_{i\to\infty} d^p(z,z), \end{aligned}$$

implies that $d^p(z,\Gamma z) = 0$, i.e., $\gamma(\Gamma z) \leq \gamma(z)$. by uniqueness of asymptotic centre, we have $\Gamma z = z$.

For uniqueness of Γ, let $\nu_i = \kappa$ and $\nu_j = z$, using inequality (2.6) and Lemma 2.6, we have

$$d^p(\kappa_i, \lambda_2\kappa \oplus \lambda_1 z) \quad \leq \quad \lambda_2 d^p(\kappa_i,\kappa) + \lambda_1 d^p(z,\kappa_i) - \lambda_1\lambda_2 d^p(\kappa,z).$$

Taking the limit on both sides of the above inequality, we obtain

$$\begin{aligned} \limsup_{i\to\infty} d^p(\kappa_i, \lambda_2\kappa \oplus \lambda_1 z) \quad \leq \quad & \limsup_{i\to\infty} \lambda_2 d^p(\kappa_i,\kappa) + \\ & \limsup_{i\to\infty} \lambda_1 d^p(z,\kappa_i) - \limsup_{i\to\infty} \lambda_1\lambda_2 d^p(\kappa,z). \end{aligned}$$

Since

$$\limsup_{i\to\infty} d^p(\kappa_i, z) \quad \leq \quad \limsup_{i\to\infty} d^p(\kappa_i \lambda_2 \kappa \oplus \lambda_1 z).$$

Implies that

$$\begin{aligned}
\limsup_{i\to\infty} d^p(\kappa_i, z) &\leq \limsup_{i\to\infty} \lambda_2 d^p(\kappa_i, \kappa) + \\
&\quad \limsup_{i\to\infty} \lambda_1 d^p(z, \kappa_i) - \limsup_{i\to\infty} \lambda_1\lambda_2 d^p(\kappa, z). \\
\limsup_{i\to\infty} d^p(z, z) &\leq \limsup_{i\to\infty} \lambda_2 d^p(z, \kappa) + \\
&\quad \limsup_{i\to\infty} \lambda_1 d^p(z, z) - \limsup_{i\to\infty} \lambda_1\lambda_2 d^p(\kappa, z). \\
0 &\leq \limsup_{i\to\infty} \lambda_2 d^p(z, \kappa) - \limsup_{i\to\infty} \lambda_1\lambda_2 d^p(\kappa, z). \\
0 &\leq \limsup_{i\to\infty} \lambda_1\lambda_2 d^p(\kappa, z). \\
\limsup_{i\to\infty} \lambda_1\lambda_2 d^p(\kappa, z) &\geq 0, \\
\limsup_{i\to\infty} d^p(\kappa, z) &\geq 0,
\end{aligned}$$

which is a contradiction. Hence $\kappa = z$. Therefore z is a unique fixed point of Γ. This argument is equivalent to Opial's condition. Q.E.D.

The second main results, consider analog of theorem in the setting of an extended interpolative Hardy-Rogers type unifomly convex contraction mapping by extending Defintion 2.14 and Theorem 2.16.

Definition 3.2. A self mapping Γ on X is said to be $((\lambda_1, \lambda_2), p)$-extended interpolative contraction, if for some $\lambda_i \in (0,1)$ and $p \geq 1$, there exists $\beta_i \geq 0$ and $\lambda_1 + \lambda_2 = 1$ satisfying the following inequality

$$\begin{aligned}
\lambda_1 d^p(\Gamma\kappa, \Gamma\nu) + \lambda_2 d^p(\Gamma^2\kappa, \Gamma^2\nu) \quad &\leq \quad \beta_1 d^p(\kappa, \nu) + \beta_2 d^p(\kappa, \Gamma\kappa) \\
&\quad +\beta_3 d^p(\Gamma\kappa, \Gamma^2\kappa) + \beta_4 d^p(\nu, \Gamma\nu) \\
&\quad +(1 - (\beta_1 + \beta_2 + \beta_3 + \beta_4)) d^p(\nu, \Gamma^2\nu), \qquad (3.10)
\end{aligned}$$

$\forall\ \kappa, \in K$.

Theorem 3.2. Let K be a non-empty bounded closed extended interpolative convex subset of a $CAT_p(0)$-metric spaces and $\Gamma : K \to K$ be $((\lambda_1, \lambda_2), p)$-convex contraction such that $\sum_{i=1}^4 \beta_i + \lambda_1 + \lambda_2 < 1$. Then, Γ has the approximated fixed point property. Further, if (X, d) is a complete $CAT_p(0)$-metric spaces, then Γ has a unique fixed point.

Proof. For any $\kappa_0 \in X$, we construct a sequence $\{\kappa_n\}$ by $\kappa_i = \Gamma^i \kappa_0$ for each $i \in \mathbb{N}$. If there exists i such that $\kappa_i = \kappa_{i+1}$, then κ_i is a fixed pont of Γ. Then proof is completed. Otherwise, assume that $\kappa_i \neq \kappa_{i+1}$ for each $i \geq 0$. By substituting $\kappa = \kappa_0$ and $\nu = \Gamma\kappa_0$ in (3.11), we find that

$$\begin{aligned} \lambda_1 d^p(\Gamma\kappa_0,\Gamma^2\kappa_0)+\lambda_2 d^p(\Gamma^2\kappa_0,\Gamma^3\kappa_0) &\leq \beta_1 d^p(\kappa_0,\Gamma\kappa_0)+\beta_2 d^p(\kappa_0,\Gamma\kappa_0)\\ &\quad+\beta_3 d^p(\Gamma\ \kappa_0,\Gamma^2\kappa_0)+\beta_4 d^p(\Gamma\kappa_0,\Gamma^2\kappa_0)\\ &\quad+(1-(\beta_1+\beta_2+\beta_3+\beta_4))d^p(\Gamma\kappa_0,\Gamma^3\kappa_0),\\ \lambda_1 d^p(\Gamma\kappa_0,\Gamma^2\kappa_0) &\leq (\beta_1+\beta_2)d^p(\kappa_0,\Gamma\kappa_0)+(\beta_3+\beta_4)d^p(\Gamma\kappa_0,\Gamma^2\kappa_0)\\ &\quad+(\lambda_2+(1-(\beta_1+\beta_2+\beta_3+\beta_4)))d^p(\Gamma\kappa_0,\Gamma^3\kappa_0). \end{aligned}$$

Using (3.3) in the above inequality, we obtain

$$\begin{aligned} \lambda_1 d^p(\Gamma\kappa_0,\Gamma^2\kappa_0) &\leq (\beta_1+\beta_2)d^p(\kappa_0,\Gamma\kappa_0)+(\beta_3+\beta_4)d^p(\Gamma\kappa_0,\Gamma^2\kappa_0)\\ &\quad+(\lambda_2+(1-(\beta_1+\beta_2+\beta_3+\beta_4)))\lambda_1 d^p(\kappa_0,\Gamma\kappa_0)\\ &\quad+\lambda_2(\lambda_2+(1-(\beta_1+\beta_2+\beta_3+\beta_4)))d^p(\Gamma\kappa_0,\Gamma^2\kappa_0),\\ (\lambda_1-[\lambda_2(\lambda_2+1)&-\lambda_2(\beta_1+\beta_2)+(1-\lambda_2)(\beta_3+\beta_4)])d^p(\Gamma\kappa_0,\Gamma^2\kappa_0)\leq\\ &\lambda_2(\lambda_2+1)+\lambda_2(\beta_1+\beta_2)-\lambda_1(\beta_3+\beta_4)d^p(\kappa_0,\Gamma\kappa_0),\\ d^p(\Gamma\kappa_0,\Gamma^2\kappa_0) &\leq \frac{\lambda_2(\lambda_2+1)+\lambda_2(\beta_1+\beta_2)-\lambda_1(\beta_3+\beta_4)}{(\lambda_1-[\lambda_2(\lambda_2+1)-\lambda_2(\beta_1+\beta_2)+(1-\lambda_2)(\beta_3+\beta_4)])}d^p(\kappa_0,\Gamma\kappa_0) \end{aligned}$$

Let $\delta=\frac{\lambda_2(\lambda_2+1)+\lambda_2(\beta_1+\beta_2)-\lambda_1(\beta_3+\beta_4)}{(\lambda_1-[\lambda_2(\lambda_2+1)-\lambda_2(\beta_1+\beta_2)+(1-\lambda_2)(\beta_3+\beta_4)])}$. We have

$$d^p(\Gamma\kappa_0,\Gamma^2\kappa_0)\leq \delta^i d^p(\kappa_0,\Gamma\kappa_0) \tag{3.11}$$

Leads to

$$\lim_{i\to\infty} d^p(\Gamma\kappa_0,\Gamma^2\kappa_0)=0. \tag{3.12}$$

which is a contradiction. The other steps to show tha $\{\kappa_i\}$ is a Cauchy sequence and existence of fixed point and its uniqueness follow similarly proof of Theorem 3.1. This complete the proof.

Q.E.D.

For the useful of Theorem 3.1, we prove the following corollary using the concepts from Dass and Gupta [29] (Theorem 2.20) and Mohammadi *et al.* [28] (Theorem 2.19).

Corollary 3.3. Let K be a non-empty bounded closed extended interpolative convex subset of a $CAT_p(0)$-metric spaces and $\Gamma: K\to K$ be $((\lambda_1,\lambda_2),p)$-convex contraction such that $\sum_{i=1}^{2}\beta_i+\lambda_1+\lambda_2<1$, such that

$$\begin{aligned} \lambda_1 d^p(\kappa,\nu)+\lambda_2\frac{d^p(\nu,T\nu)[1+d^p(\kappa,\Gamma\kappa)]}{1+d^p(\kappa,\nu)} &\leq \beta_1 d^p(\kappa,\nu)+\beta_2 d^p(\kappa,\Gamma\kappa)\\ &\quad+(1-(\beta_1+\beta_2))d^p(\nu,\Gamma\nu), \end{aligned} \tag{3.13}$$

$\forall\ \kappa,\in K$.

Then, Γ has the approximated fixed point property. Further, if (X,d) is a complete $CAT_p(0)$-metric spaces, then Γ has a unique fixed point.

Proof. Following the lines in proof of Theorem 3.1, let $\kappa_0 \in X$, define a picard sequence $\{\kappa_n\}$ by $\kappa_i = \Gamma^i \kappa_0$ for each $i \in \mathbb{N}$. Taking $\kappa = \kappa_0$ and $\nu = \Gamma\kappa_0$ in (3.11), gives

$$\begin{aligned}
\lambda_1 d^p(\kappa_0, \Gamma\kappa_0) + \lambda_2 \frac{d^p(\Gamma\kappa_0, T^2\kappa_0)[1 + d^p(\kappa_0, \Gamma\kappa_0)]}{1 + d^p(\kappa_0, \Gamma\kappa_0)} &\leq \beta_1 d^p(\kappa_0, \Gamma\kappa_0) + \beta_2 d^p(\kappa_0, \Gamma\kappa_0) \\
&\quad +(1 - (\beta_1 + \beta_2))d^p(\Gamma\kappa_0, \Gamma^2\kappa_0), \\
\lambda_1 d^p(\kappa_0, \Gamma\kappa_0) + \lambda_2 d^p(\Gamma\kappa_0, T^2\kappa_0) &\leq (\beta_1 + \beta_2)d^p(\kappa_0, \Gamma\kappa_0) \\
&\quad +(1 - (\beta_1 + \beta_2))d^p(\Gamma\kappa_0, \Gamma^2\kappa_0), \\
(\lambda_2 - (1 - (\beta_1 + \beta_2)))d^p(\Gamma\kappa_0, T^2\kappa_0) &\leq ((\beta_1 + \beta_2) - \lambda_1)d^p(\kappa_0, \Gamma\kappa_0), \\
d^p(\Gamma\kappa_0, T^2\kappa_0) &\leq \frac{((\beta_1 + \beta_2) - \lambda_1)}{(\lambda_2 - (1 - (\beta_1 + \beta_2)))} d^p(\kappa_0, \Gamma\kappa_0),
\end{aligned}$$

Again by taking $\vartheta = \frac{((\beta_1+\beta_2)-\lambda_1)}{(\lambda_2-(1-(\beta_1+\beta_2)))}$, we obtain

$$d^p(\Gamma\kappa_0, T^2\kappa_0) \leq \vartheta d^p(\kappa_0, \Gamma\kappa_0),$$

By induction, we get

$$d^p(\Gamma\kappa_0, T^2\kappa_0) \leq \vartheta^i d^p(\kappa_0, \Gamma\kappa_0),$$

which implies that

$$\lim_{i\to\infty} d^p(\Gamma\kappa_0, T^2\kappa_0) = 0,$$

which is a contradiction. The prove of other steps follow the similar proof of Theorem 3.1. This completes the proofs. Q.E.D.

Next, an example is given below for supporting the proven results above.

Example 3.4. Let $X = [0, 2]$ and $K = \{0, 1\} \in X$ and (X, d) be a complete $CAT_p(0)$-metric spaces, $p \geq 2$. Define a metric by $d(\kappa, \nu) = \|\kappa - \nu\|$, and a mapping $\Gamma : X \to X$ given by

$$\Gamma\kappa = \frac{1 - \kappa^2}{2}.$$

We show that Γ satisfy inequality (2.6) and (3.1).

First we calculate the following metrics,

$$\begin{aligned}
d^p(\kappa, \nu) &= \|\kappa - \nu\|^p, \\
d^p(\Gamma\kappa, \Gamma\nu) = d^p\Big(\frac{1 - \kappa^2}{2}, \frac{1 - \nu^2}{2}\Big) &= \Big|\Big|\frac{1 - \kappa^2}{2} - \frac{1 - \nu^2}{2}\Big|\Big|^p = \Big|\Big|\frac{\nu^2 - \kappa^2}{2}\Big|\Big|^p, \\
d^p(\Gamma^2\kappa, \Gamma^2\nu) = d^p\Big(\Big(\frac{1 - \kappa^2}{2}\Big)^2, \Big(\frac{1 - \nu^2}{2}\Big)^2\Big) &= \Big|\Big|\Big(\frac{1 - \kappa^2}{2}\Big)^2 - \Big(\frac{1 - \nu^2}{2}\Big)^2\Big|\Big|^p \\
&= \Big|\Big|\frac{\kappa^4 - \nu^4 - 2\kappa^2 - 2\nu^2}{4}\Big|\Big|^p,
\end{aligned}$$

By applying all above equalities in (3.1), $\kappa = 0, \nu = 1$ with $\lambda_1 = \frac{1}{4}, \lambda_2 = \frac{3}{4}$, we get

$$\begin{aligned}
\lambda_1\Big|\Big|\frac{\nu^2-\kappa^2}{2}\Big|\Big|^p + \lambda_2\Big|\Big|\frac{\kappa^4-\nu^4-2\kappa^2-2\nu^2}{4}\Big|\Big|^p &\leq \beta\|\kappa-\nu\|^p, \\
\frac{1}{4}\Big|\Big|\frac{1^2-0^2}{2}\Big|\Big|^p + \frac{3}{4}\Big|\Big|\frac{0^4-1^4-2.0^2-2^2}{4}\Big|\Big|^p &\leq \beta\|0-1\|^p, \\
\frac{1}{4}\Big|\Big|\frac{1}{2}\Big|\Big|^p + \frac{3}{4}\Big|\Big|\frac{-3}{4}\Big|\Big|^p &\leq \beta\|0-1\|^p, \\
\frac{1}{2}^{p+2} + \frac{3}{4}^{p+1} &\leq \beta,
\end{aligned}$$

by schoosing the appropriate value of β and taking $p \geq 2$ in the above inequality, the condition imposed in Theorem 3.1 are satisfied.

Similarly,

$$d^p(\lambda_2\kappa \oplus \lambda_1\nu, z) \leq \lambda_2 d^p(\kappa, z) + \lambda_1 d^p(\nu, z) - \lambda_1\lambda_2 d^p(\kappa, \nu). \tag{3.14}$$

where

$$d^p(\lambda_2\kappa \oplus \lambda_1\nu, z) \leq \lambda_2 d^p(\kappa, z) + \lambda_1 d^p(\nu, z), \tag{3.15}$$

using (3.15) in (3.14), we get

$$\begin{aligned}
\lambda_2 d^p(\kappa, z) + \lambda_1 d^p(\nu, z) &\leq \lambda_2 d^p(\kappa, z) + \lambda_1 d^p(\nu, z) - \lambda_1\lambda_2 d^p(\kappa, \nu), \\
0 &\leq -\lambda_1\lambda_2 d^p(\kappa, \nu). \\
0 &\leq \lambda_1\lambda_2\|\kappa-\nu\|^p, \\
0 &\leq \frac{1}{4}.\frac{3}{4}\|0-1\|^p, \\
0 &\leq \frac{3}{16}.
\end{aligned}$$

Hence, Γ satisfy the condition imposed in X as a $CAT_p(0)$ metric space. Thus Γ have an approximated fixed point $\kappa = -1 + \sqrt{2}$.

4 Existence of the solution to Hopf bifurcations in a delayed-energy based model of capital accumulation in $CAT_p(0)$ metric space

This section investigate the Existence of the Solution to Hopf bifurcations in a delayed-energy based model of capital accumulation in $CAT_p(0)$ metric space for demonstration of Theorem 3.1. In modelling of the economic growth depends on the capability to exploit the increasing amount of energy. According to the quantity of energy that society consumes becomes an economic tool to measure its progress. This study is motivated by Dalgaard and Strulik model [30], which shows a relationship between a concave and long-linear Kleiber's law relation, between electricity consumption per capital. On the other hand it concerned with the modelling of an economy viewed as transportation network of electricity. Electricity is used to run, maintain, and create capital. Assuming that time is continuous, and let μ be energy required to operate and v is the energy costs to create a new capital good. Energy conservation implies

$$e(t) \quad = \quad \mu\gamma(t) + v\frac{d\gamma(t)}{dt}. \tag{4.1}$$

The mathematical model is derived by modelling the energy as

$$e(t) \quad = \quad \varepsilon[\gamma(t)]^a, \tag{4.2}$$

where $0 < a < 1$ is the real constant proportional to the dimension and efficiency of the network, and $\varepsilon > 0$ is real constant in the sense that it is independent of capital per worker. The model becomes

$$\varepsilon[\gamma(t)]^a \quad = \quad \mu\gamma(t) + v\frac{d\gamma(t)}{dt}. \tag{4.3}$$

Equivalent to

$$\frac{d\gamma(t)}{dt} \quad = \quad \frac{\varepsilon}{v}[\gamma(t)]^a - \frac{\mu}{v}\gamma(t). \tag{4.4}$$

If equation (4.1) contains a time delay T as

$$e(t-T) \quad = \quad \mu\gamma(t-T) + v\frac{d\gamma(t)}{dt}. \tag{4.5}$$

Consequently, the law of motion of capital is described by the following non-linear delay differential equation

$$\frac{d\gamma(t)}{dt} \quad = \quad \frac{\varepsilon}{v}[\gamma(t-T)]^a - \frac{\mu}{v}\gamma(t-T), \tag{4.6}$$

for some initial function $\gamma(t) = \varphi(t), t \in [-T, 0]$.

Consider for zero delay $T = 0$. There exists a unique positive state $\gamma_\star$ satisfying the relation $\varepsilon\gamma_\star^{a-1} = \mu$. Setting $\kappa(t) = \gamma(t) - \gamma_\star$, then (4.6) becomes

$$\frac{d\kappa(t)}{dt} \quad = \quad \frac{\varepsilon}{v}[\kappa(t-T) + \gamma_\star]^a - \frac{\mu}{v}[\kappa(t-T) + \gamma_\star], \tag{4.7}$$

where $\gamma_\star$ is the equilibrium point.

The linearization of (4.7) at zero [31] is given by

$$\frac{d\kappa(t)}{dt} \quad = \quad \frac{a-1}{v}\mu\kappa(t-T). \tag{4.8}$$

At the zero equilibrium, using Taylor series expansion up to the second order as

$$\frac{d\kappa(t)}{dt} \quad = \quad \frac{a-1}{v}\mu\kappa(t-T) + \frac{a(a-1)}{2v}\mu\gamma_\star^{-1}[\kappa(t-T)]^2. \tag{4.9}$$

The equation (4.9) can be represented as a fixed point equation.

$$\kappa(t) \quad = \quad \int_0^T \Big(\lambda_1 \Gamma\kappa(t-T) + \lambda_2 \Gamma^2 \kappa(t-T)\Big) dt, \tag{4.10}$$

where $\lambda_1 = \frac{a-1}{v}\mu$ and $\lambda_2 = \frac{a(a-1)}{2v}\mu\gamma_\star^{-1}$.

Let $(X, \|.\|)$ be a Banach space. Define the mapping $d : X \times X \to [0, \infty)$ by

$$d^p(\kappa, \nu) = \sup_{t\in[0,T]} \|\kappa - \nu\|^p, p \geq 2.$$

Then (X, d) is a complete $CAT_p(0)$ metric space. We present the following theorem:

Theorem 4.1. Suppose the following hypotheses hold:

(i) There exists an asymptotically regular function such that

$$\lambda_1 \|\Gamma\kappa - \Gamma\nu\|^p + \lambda_2 \|\Gamma^2\kappa - \Gamma^2\nu\|^p \quad \leq \quad \beta\|\kappa - \nu\|^p,$$

where

$$\lambda_1 d^p(\Gamma\kappa, \Gamma\nu) + \lambda_2 d^p(\Gamma^2\kappa, \Gamma^2\nu) \quad \leq \quad \beta d^p(\kappa.\nu)$$

(ii) there exists a constant β such that

$$\beta \leq \frac{1}{T},$$

$T \in [0, 1]$.

Then, the integral equation (4.10) has a unique positive equilibrium point which undergoes a Hupf bifurcation at κ.

Proof. From condition (i) and (ii), it is known that the stability of the positive steady state and local Hopf bifurcations can be determined by the distribution of the roots associated with the characteristic equation associated by linearization. Now, we show that integral equation (4.10) can attain a unique positive equilibrium point which undergoes a Hupf bifurcation at κ.

Define a mapping $\Gamma : C^1([0, T], X) \to C^1([0, T], X)$ in X as in (4.10).

By continuity property, for $0 \leq t \leq T$, we have $\Gamma\kappa, \Gamma\nu \in X$ and $\Gamma\ \kappa \neq \Gamma\nu$. Let $\kappa, \nu \in C^1([0, T], X)$

fort $\nu \leq \kappa$, we get

$$\begin{aligned}
\|\kappa(t-T)-\nu(t-T)\|^p &= \Big\|\int_0^T \Big(\lambda_1\Gamma\kappa(t-T)+\lambda_2\Gamma^2\kappa(t-T)\Big)dt - \\
&\quad \int_0^T \Big(\lambda_1\Gamma\nu(t-T)+\lambda_2\Gamma^2\nu(t-T)\Big)dt\Big\|, \\
\|\kappa(t-T)-\nu(t-T)\|^p &\geq \Big(\lambda_1\|\Gamma\kappa(t-T)-\Gamma\nu(t-T)\|^P + \\
&\quad \lambda_2\|\Gamma^2\kappa(t-T)-\Gamma^2\nu(t-T)\|^p\Big)\int_0^T dt, \\
\|\kappa(t-T)-\nu(t-T)\|^p &\geq \Big(\lambda_1\|\Gamma\kappa(t-T)-\Gamma\nu(t-T)\|^P + \\
&\quad \lambda_2\|\Gamma^2\kappa(t-T)-\Gamma^2\nu(t-T)\|^p\Big)T, \\
\frac{1}{T}\|\kappa(t-T)-\nu(t-T)\|^p &\geq \lambda_1\|\Gamma\kappa(t-T)-\Gamma\nu(t-T)\|^P + \\
&\quad \lambda_2\|\Gamma^2\kappa(t-T)-\Gamma^2\nu(t-T)\|^p, \\
\beta d^p(\kappa(t-T),\nu(t-T)) &\geq \lambda_1 d^p(\Gamma\kappa(t-T),\Gamma\nu(t-T)) + \\
&\quad \lambda_2 d^p(\Gamma^2\kappa(t-T),\Gamma^2\nu(t-T)),
\end{aligned}$$

which implies

$$\lambda_1 d^p(\Gamma\kappa(t-T),\Gamma\nu(t-T)) + \lambda_2 d^p(\Gamma^2\kappa(t-T),\Gamma^2\nu(t-T)) \leq \beta d^p(\kappa(t-T),\nu(t-T)),$$

which is a contradiction. Hence, the integral equation (4.10) has a unique positive equilibrium point which undergoes a Hupf bifurcation at κ. Q.E.D.

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Bibliography

[1] M. Gromov. *Metric structure for Riemannian and Non-Riemannian spaces,* Progr. Math., **152** Birkhauser, Boston,1984.

[2] S. Dhompongsa and B. Panyanak. *On △-convergence theorems in CAT (0) spaces,* Computers and Mathematics with Applications., **56**(10) (2008), 2572-2579.

[3] S. Reich and I. Shafrir, *Nonexpansive iterations in hyperbolic spaces,* Nonlinear analysis: theory, methods and applications **15**(6) (1990), 537-558.

[4] K. Goebel and W.A. Kirk. *Iteration processes for nonexpansive mappings,* Topological Methods in Nonlinear Functional Analysis. Contemporary Mathematics AMS,, **21** (1983), 115–123.

[5] K. Goebel and S. Reich. *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings,* Series of Monographs and Textbooks in Pure and Applied Mathematics. Dekker, New York, **1984**.

[6] W.A. Kirk. *A fixed point theorem for mappings which do not increases distances,* Am. Math. Mon., **72** (1965), 1004–1006.

[7] U. Kohlenbach and L. Leustean. *Mann iterates of directionally nonexpansive mappings in hyperbolic spaces,* Abstr. Appl. Anal., **8** (2003), 449–477.

[8] B. Nanjaras, B. Panyanaka and W. Phuengrattanab. *Fixed point theorems and convergence theorems for Suzuki-generalized nonexpansive mappings in C AT (0) spaces,* Nonlinear Anal. Hybrid Syst., **4** (2010), 25–31.

[9] S. Shukri. *On monotone nonexpansive mappings in* $CAT_p(0)$ *spaces,* Fixed Point Theory Appl., **8** (2020). https://doi. org/10.1186/s13663-020-00675-z

[10] A. Papadopoulos. *Metric Spaces, Convexity and Nonpositive Curvature, IRMA Lectures in Mathematics and Theoretical Physics,* **vol. 6** European Mathematical Society (EMS), Zurich, 2005).

[11] W.A. Kirk. *Fixed point theorems in CAT(0) spaces and R-trees,* Fixed Point Theory Appl., **2004** (2004), 309–316.

[12] M.A. Khamsi and S. Shukri. *Generalized CAT(0) spaces,* Bull. Belg. Math. Soc. Simon Stevin, **24** (2017), 417–426.

[13] A.A. Darweesh and S. Shukri. *Fixed points of Suzuki-generalized nonexpansive mappings in* $CAT_p(0)$ *metric spaces,* Arabian Journal of Mathematics, (2024), 1–10.

[14] R.P. Agarwal, J. Mohamed and B. Samet. *Fixed Point Theory in Metric Spaces,* in Recent Advances and Applications, Springer, 2018.

[15] F. Bruhat and J. Tits. *Groupes réductifs sur un corps local. I. Données radicielles valuées,* Inst. Hautes Études Sci. Publ. Math, **41** (1972), 5–251.

[16] K. Calderon, M.A. Khamsi and J. Martínez-Moreno. *Perturbed approximations of fixed points of nonexpansive mappings in* $CAT_p(0)$ *spaces,* Carpathian Journal of Mathematics., **37**(1) (2021), 65-79.

[17] A. Naor and L. Silberman *Poincare inequalities, embeddings, and wild groups,* Compos. Math., **147**(5) (2011), 1546–1572.

[18] K. Goebel and M. Japon-Pineda. *A new type of nonexpansiveness,* Proc. of the 8th International Conference on Fixed Point Theory and Appl. Chiang Mai., (2007).

[19] M. Asadi, S. Ghasemzadehdibagi, S. Haghayeghi and N. Ahmad. *Fixed point theorem for* (α, p)*-nonexpansive in CAT (0) spaces,* Nonlinear Functional Analysis and Applications, (2021), 1045-1057

[20] M.S. Khan, Y.M. Singh, G. Maniu and M. Postolache. *On* (α, p)*-convex contraction and asymptotic regularity,* J. Math. Comput. Sci,, **18**, (2018), 132-145.

[21] M. Berinde. *Approximate fixed point theorems,* Stud. Univ. Babes¸-Bolyai Math., **51**(1),(2006), 11–25.

[22] F. E. Browder, W. V. Petryshyn. *The solution by iteration of nonlinear functional equations in Banach spaces,* Bull. Amer. Math. Soc., **72**,(1996) 571–575. 1.4

[23] V.I. Istratescu, V. I. *Some fixed point theorems for convex contraction mappings and convex nonexpansive mappings,* Libertas Mathematica, **1**, (1981), 151-164.

[24] Z. Opial. *Weak convergence of the sequence of successive approximations for nonexpansive mappings,* Bull. Amer. Math. Soc., **73**(35), (1967), 591-597.

[25] F.E. Browder. *Nonlinear operators and nonlinear equations of evolution in Banach spaces,* In Proc. Symp. Pure Math.. Amer. Math. Soc., **18**(2),(1976), 591-597.

[26] E. Karapinar. *Revisiting the Kannan type contractions via interpolation,* Adv. Theory Nonlinear Anal. Appl., **2** (2018), 85-87.

[27] E. Karapinar, 0. Alqahtani and H. Aydi. *On interpolative Hardy-Rogers type contractions,* Symmetry, **11**(1) (2019), 8.

[28] Mohammadi B, Parvaneh V, Aydi H. On extended interpolative Ćirić-Reich-Rus type F-contractions and an application, *Journal of Inequalities and Applications,* **2019** (2019), 1-11.

[29] B.K Dass and s. Gupta. *An extension of the Banach contraction principle through rational expression,* Indian J. pure appl. Math, **6**(1975) 1455-1458.

[30] C.L Dalgaard and H. Strulik. *Energy distribution and economic growth,* Resource and Energy Economics, **33**(2011), 782-797.

[31] S. Ruan and J. Wei. *On the zeros of transcendental functions with applications to stability of delay differential equations with two delays, ynamics of Continuous,* Discrete and Impulsive Systems. Series A. Mathematical Analysis, **10**(2003), 863-874.

[32] T. Suzuki. *Fixed point theorems and convergence theorems for some generalized nonexpansive mapping,* J. Math. Anal., **340**(2), (2008), 1088–1095.