Asymptotically lacunary statistical equivalence set sequences of order (α, β)

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Abstract

In this paper we define asymptotically lacunary statistical equivalence set sequences of order (α, β) and studied some properties of this concept. We also make an effort to define strongly asymptotically lacunary p-equivalence and strongly Cesàro asymptotically p-equivalence of order (α, β) and examine some topological properties.

2010 Mathematics Subject Classification. **40A35**. 40G15, 40C05.
Keywords. lacunary sequence, statistical convergence, Cesàro summability, asymptotically equivalence, sequences of sets.

1 Introduction and preliminaries

Numerous authors have expanded the idea of convergence of sequences of numbers to include sequences of sets. The idea of Wijsman convergence is one of these extensions that is taken into consideration in this paper (see[3,4,10,11,13,14,16]) and references therein. By extending the idea of set sequence convergence to statistical convergence, Nuray and Rhoades[16] provided some fundamental theorems. It should be mentioned that Fridy and Orhan examined lacunary statistical convergence ([5]). Wijsman lacunary statistical convergence of sequence of sets was defined by Ulusu and Nuray [9] and its connection to Wijsman statistical convergence was investigated.
Definitions of asymptotically equivalent sequences and asymptotic regular matrices were provided by Marouf[7]. By providing an analogue of these definitions and natural regularity criteria for nonnegative summabilitry matrices, Patterson [15] expanded on these ideas. The concepts provided in [17] were expanded by Patterson and Savaş to include lacunary statistically comparable sequences. These definitions were accompanied by natural inclusion theorems. $\mathcal{I}$-asymptotically lacunary statistical equivalent sequences were presented by Savaş[2]. Ulusu and Nuray[9] also expanded the concepts given in [17] to include sequence of sets, which is in Wijsman sense. For more details about sequence spaces of order (α, β) see ([1],[6],[8],[12],[18],[19]) and references therein.

Definition 1.1. Suppose that $z = (z_n)$ and $t = (t_n)$ are any two nonnegative sequences then (z_n) and (t_n) are asymptotically equivalent if

$$\lim_n \frac{z_n}{t_n} = 1$$

(represented by $z \sim t$).

Consider a metric space $(Z, \mathcal{J})$. For $z \in Z$ and A be any non-empty subset of Z, the distance between z and A is determined by

$$\delta(z, A) = \inf_{u \in A} \mathcal{J}(z, u).$$

Advanced Studies: Euro-Tbilisi Mathematical Journal 16(3) (2023), pp. 67–75.
DOI: 10.32513/asetmj/193220082326
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 08 March 2023.
Accepted for publication: 28 May 2023.

Definition 1.2. Consider a metric space $(Z, \mathcal{J})$. For A, A_k be any closed non-empty subsets of Z, then the sequence $\{A_k\}$ is Wijsman convergent to A if

$$\lim_{k\to\infty} \delta(z, A_k) = \delta(z, A), \text{ for each } z \in Z.$$

Specifically, we write it as $\lim_W A_k = A$.

Definition 1.3. The sequence $z = (z_k)$ is known to be statistically convergent to L of order (α, β) if for every $\varepsilon > 0$,

$$\lim_m \frac{1}{m^\alpha} |\{k \le m : |z_k - L| \ge \varepsilon\}|^\beta = 0$$

(represented by $st - \lim z_{k(\alpha,\beta)} = L$).

Definition 1.4. The sequence $z = (z_k)$ is known to be strongly Cesàro summable to the number L of order (α, β) if $\lim_{m\to\infty} \frac{1}{m^\alpha} \left[\sum_{k=1}^m |z_k - L|\right]^\beta = 0$

(represented by $z_{k(\alpha,\beta)} \xrightarrow{|\sigma_1|} L$)

Definition 1.5. Consider a metric space $(Z, \mathcal{J})$. For A, A_k be any closed non-empty subsets of Z, then $\{A_k\}$ is Wijsman statistical convergent to A of order (α, β) if $\{\delta(z, A_k)\}$ is statistically convergent to $\delta(z, A)$ of order (α, β); i.e., for $\varepsilon > 0$ and for each $z \in Z$,

$$\lim_{m\to\infty} \frac{1}{m^\alpha} |\{k \le m : |\delta(z, A_k) - \delta(z, A)| \ge \varepsilon\}|^\beta = 0$$

we write it as $st - \lim_W A_{k(\alpha,\beta)} = A$ or $A_{k(\alpha,\beta)} \to A(WS)$.

Definition 1.6. Consider a metric space $(Z, \mathcal{J})$. For A, A_k be any closed non-empty subsets of Z, then $\{A_k\}$ is Wijsman strongly Cesàro summable to A of order (α, β) if for each $z \in Z$,

$$\lim_{m\to\infty} \frac{1}{m^\alpha} \left[\sum_{k=1}^m |\delta(z, A_k) - \delta(z, A)|\right]^\beta = 0.$$

We write it as $A_{k(\alpha,\beta)} \to A([W\sigma_1])$ or $A_{k(\alpha,\beta)} \xrightarrow{[W\sigma_1]} A$.

A lacunary sequence $\theta = \{k_r\}$ is an increasing integer sequence such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. In this paper, by θ we mean the intervals and it will be represented by $I_r = (k_{r-1}, k_r]$, and the ratio $\frac{k_r}{k_{r-1}}$ is reduced to q_r.

Definition 1.7. Consider a metric space $(Z, \mathcal{J})$, $0 < \alpha \le \beta \le 1$ and a lacunary sequence $\theta = \{k_r\}$. For A, A_k be any closed non-empty subsets of Z , then $\{A_k\}$ is Wijsman lacunary statistical convergent to A of order (α, β) if $\{\delta(z, A_k)\}$ is lacunary statistically convergent to $\delta(z, A)$; i.e., for $\varepsilon > 0$ and for each $z \in Z$,

$$\lim_r \frac{1}{h_r^\alpha} |\{k \in I_r : |\delta(z, A_k) - \delta(z, A)| \ge \varepsilon\}|^\beta = 0.$$

Specifically, we write it as $S_\theta - \lim_W A_{k(\alpha,\beta)} = A$ or $A_{k(\alpha,\beta)} \to A(WS_\theta)$.

Definition 1.8. Consider a metric space $(Z, \mathcal{J})$, $0 < \alpha \leq \beta \leq 1$ and a lacunary sequence θ. For A_k, B_k be any closed non-empty subsets of Z such that $\delta(z, A_k) > 0$ and $\delta(z, B_k) > 0$ for each $z \in Z$, then the sequence $\{A_k\}$ and $\{B_k\}$ are asymptotically lacunary statistical equivalent of order (α, β) of multiplicity L if for every $\varepsilon > 0$ and each $z \in Z$,

$$\lim_r \frac{1}{h_r^\alpha} |\{k \in I_r : |\frac{\delta(z, A_k)}{\delta(z, B_k)} - L| \geq \varepsilon\}|^\beta = 0$$

(represented as $\{A_k\}_{(\alpha,\beta)} \overset{WS_\theta^L}{\sim} \{B_k\}_{(\alpha,\beta)}$) and simply asymptotically lacunary statistical equivalent of order (α, β) if $L = 1$.

Definition 1.9. Consider a metric space $(Z, \mathcal{J})$, $0 < \alpha \leq \beta \leq 1$ and a lacunary sequence θ. For A_k, B_k be any closed non-empty subsets of Z such that $\delta(z, A_k) > 0$ and $\delta(z, B_k) > 0$ for each $z \in Z$, then the sequence $\{A_k\}$ and $\{B_k\}$ are strongly asymptotically lacunary statistical equivalent of order (α, β) of multiplicity L if for each $z \in Z$,

$$\lim_r \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left| \frac{\delta(z, A_k)}{\delta(z, B_k)} - L \right| \right]^\beta = 0$$

(represented as $\{A_k\}_{(\alpha,\beta)} \overset{[WN]_\theta^L}{\sim} \{B_k\}_{(\alpha,\beta)}$) and simply strongly asymptotically lacunary equivalent of order (α, β) if $L = 1$.

2 Main results

Consider a metric space $(Z, \mathcal{J})$. For $A_k, B_k \subseteq Z$ be any closed non-empty subsets, we define $\delta(z; A_k, B_k)$ as:

$$\delta(z; A_k, B_k) = \begin{cases} \frac{\delta(z, A_k)}{\delta(z, B_k)}, z \notin A_k \cup B_k, \\ L \qquad , z \in A_k \cup B_k. \end{cases}$$

Definition 2.1. Consider a metric space $(Z, \mathcal{J})$, $0 < \alpha \leq \beta \leq 1$ and a lacunary sequence θ and a positive real number sequence $p = (p_k)$. For A_k, B_k be any closed non-empty subsets of Z , then the sequences $\{A_k\}$ and $\{B_k\}$ are strongly asymptotically lacunary p-equivalent of order (α, β) of multiplicity L if for each $z \in Z$,

$$\lim_r \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} |\delta(z; A_k, B_k) - L|^{p_k} \right]^\beta = 0$$

(represented as $\{A_k\}_{(\alpha,\beta)} \overset{[WN]_\theta^{L(p)}}{\sim} \{B_k\}_{(\alpha,\beta)}$) and simply strongly asymptotically lacunary p-equivalent of order (α, β) if $L = 1$.

If $p = (p_k)$ for all $k \in \mathbb{N}$, then

$\{A_k\}_{(\alpha,\beta)} \overset{[WN]_\theta^{Lp}}{\sim} \{B_k\}_{(\alpha,\beta)}$ rather than $\{A_k\}_{(\alpha,\beta)} \overset{[WN]_\theta^{L(p)}}{\sim} \{B_k\}_{(\alpha,\beta)}$.

Theorem 2.2. Consider a metric space $(Z,\mathcal{J})$ and $\theta=\{k_r\}$ be a lacunary sequence and A_k, B_k be closed non-empty subsets of Z, and $0<\alpha\leq\beta\leq 1$, then

(a) $\{A_k\}_{(\alpha,\beta)} \overset{[WN]_\theta^{L_p}}{\sim} \{B_k\}_{(\alpha,\beta)}$ implies $\{A_k\}_{(\alpha,\beta)} \overset{WS_\theta^L}{\sim} \{B_k\}_{(\alpha,\beta)}$,

(b) $\delta(z,_k)=\bigcirc(\delta(z,B_k))$ and $\{A_k\}_{(\alpha,\beta)} \overset{WS_\theta^L}{\sim} \{B_k\}_{(\alpha,\beta)}$ implies $\{A_k\}_{(\alpha,\beta)} \overset{[WN]_\theta^{L_p}}{\sim} \{B_k\}_{(\alpha,\beta)}$.

Proof. (a) Let $\varepsilon>0$ and $\{A_k\}_{(\alpha,\beta)} \overset{[WN]_\theta^{L_p}}{\sim} \{B_k\}_{(\alpha,\beta)}$.
Thus, we can write

$$\lim_r \frac{1}{h_r^\alpha}\left[\sum_{k\in I_r}|\delta(z;A_k,B_k)-L|^p\right]^\beta$$

$$\begin{aligned}
&\geq \lim_r \frac{1}{h_r^\alpha}\left[\sum_{\substack{k\in I_r\\ |\delta(z;A_k,B_k)-L|\geq\varepsilon}}|\delta(z;A_k,B_k)-L|^p\right]^\beta\\
&\geq \varepsilon^p.\lim_r \frac{1}{h_r^\alpha}|\{k\in I_r:|\delta(z;A_k,B_k)-L|\geq\varepsilon\}|^\beta.
\end{aligned}$$

Thus, $\{A_k\}_{(\alpha,\beta)} \overset{WS_\theta^L}{\sim} \{B_k\}_{(\alpha,\beta)}$.

(b) Assume that $\delta(z,A_k)=\bigcirc(\delta(z,B_k))$ and $\{A_k\}_{(\alpha,\beta)} \overset{WS_\theta^L}{\sim} \{B_k\}_{(\alpha,\beta)}$.
Consequently, we may assume that $|\delta(z;A_k,B_k)-L|\leq Q$, for all k and for each $z\in Z$.
For given $\varepsilon>0$ and N_ε be such that

$$\frac{1}{h_r^\alpha}\left|\left\{k\in I_r:|\delta(z;A_k,B_k)-L|\geq\left(\frac{\varepsilon}{2}\right)^{\frac{1}{p}}\right\}\right|^\beta\leq\frac{\varepsilon}{2K^p},$$

for all $r>N_\varepsilon$ for each $z\in Z$.
Let $C_k=\{k\in I_r:|\delta(z;A_k,B_k)-L|\geq(\frac{\varepsilon}{2})^{\frac{1}{p}}\}$. Now, for all $r>N_\varepsilon$, we get

$$\tfrac{1}{h_r^\alpha}\left[\sum_{k\in I_r}|\delta(z;A_k,B_k)-L|^p\right]^\beta$$

$$\begin{aligned}
&= \frac{1}{h_r^\alpha}\left[\sum_{k\in C_k}|\delta(z;A_k,B_k)-L|^p\right]^\beta+\frac{1}{h_r^\alpha}\left[\sum_{k\notin C_k}|\delta(z;A_k,B_k)-L|^p\right]^\beta\\
&\geq \frac{1}{h_r^\alpha}\frac{h_r^\alpha\varepsilon}{2K^p}K^p+\frac{1}{h_r^\alpha}h_r^\alpha\frac{\varepsilon}{2}.
\end{aligned}$$

Hence, $\{A_k\}_{(\alpha,\beta)} \overset{[WN]_\theta^{L_p}}{\sim} \{B_k\}_{(\alpha,\beta)}$. Q.E.D.

Theorem 2.3. Consider a metric space $(Z,\mathcal{J})$, $0<\alpha\leq\beta\leq 1$ and A_k, B_k be closed non-empty subsets of Z. If $\theta=\{k_r\}$ be a lacunary sequence and $\sup_k p_k=H$, then

$$\{A_k\}_{(\alpha,\beta)} \overset{[WN]_\theta^{L(p)}}{\sim} \{B_k\}_{(\alpha,\beta)} \text{ implies } \{A_k\}_{(\alpha,\beta)} \overset{WS_\theta^L}{\sim} \{B_k\}_{(\alpha,\beta)}$$

Proof. Assume that $\sup_k p_k = H$ and $\{A_k\}_{(\alpha,\beta)} \overset{[WN]_\theta^{L(p)}}{\sim} \{B_k\}_{(\alpha,\beta)}$. For given $\varepsilon > 0$, we have $\frac{1}{h_r^\alpha}\left[\sum_{k\in I_r} |\delta(z;A_k,B_k) - L|^{p_k}\right]^\beta$

$$
\begin{aligned}
&= \frac{1}{h_r^\alpha}\Bigg[\sum_{\substack{k\in I_r\\ |\delta(z;A_k,B_k)-L|\geq\varepsilon}} |\delta(z;A_k,B_k) - L|^{p_k}\Bigg]^\beta \\
&+ \frac{1}{h_r^\alpha}\Bigg[\sum_{\substack{k\in I_r\\ |\delta(z;A_k,B_k)-L|<\varepsilon}} |\delta(z;A_k,B_k) - L|^{p_k}\Bigg]^\beta \\
&\geq \frac{1}{h_r^\alpha}\Bigg[\sum_{\substack{k\in I_r\\ |\delta(z;A_k,B_k)-L|\geq\varepsilon}} |\delta(z;A_k,B_k) - L|^{p_k}\Bigg]^\beta \\
&\geq \frac{1}{h_r^\alpha}\Bigg[\sum_{\substack{k\in I_r\\ |\delta(z;A_k,B_k)-L|\geq\varepsilon}} (\varepsilon)^{p_k}\Bigg]^\beta \\
&\geq \frac{1}{h_r^\alpha}\Bigg[\sum_{\substack{k\in I_r\\ |\delta(z;A_k,B_k)-L|\geq\varepsilon}} \min\{(\varepsilon)^{\inf p_k}, (\varepsilon)^H\}\Bigg]^\beta \\
&\geq \frac{1}{h_r^\alpha}|\{k\in I_r : |\delta(z;A_k,B_k) - L| \geq \varepsilon\}|^\beta . \min\{(\varepsilon)^{\inf p_k}, (\varepsilon)^H\}^\beta
\end{aligned}
$$

Hence, $\{A_k\}_{(\alpha,\beta)} \overset{WS_\theta^L}{\sim} \{B_k\}_{(\alpha,\beta)}$ Q.E.D.

Theorem 2.4. Consider a metric space $(Z,\mathcal{J})$, $0 < \alpha \leq \beta \leq 1$ and A_k, B_k be closed non-empty subsets of Z. If $\delta(z,A_k) = \bigcirc(\delta(z,B_k))$ and $0 < h = \inf_k p_k \leq \sup_k p_k = H < \infty$, then

$$\{A_k\}_{(\alpha,\beta)} \overset{WS_\theta^L}{\sim} \{B_k\}_{(\alpha,\beta)} \text{ implies } \{A_k\}_{(\alpha,\beta)} \overset{[WN]_\theta^{L(p)}}{\sim} \{B_k\}_{(\alpha,\beta)}.$$

Proof. Suppose that $\delta(z,A_k) = \bigcirc(\delta(z,B_k))$ and $\varepsilon > 0$ be given. Since $\delta(z,A_k) = \bigcirc(\delta(z,B_k))$, then there exists an integer Q such that $|\delta(z;A_k,B_k) - L| \leq Q$,
for all k and for each $z \in Z$. Then

$$\frac{1}{h_r^\alpha}\left[\sum_{k\in I_r}|\delta(z;A_k,B_k)-L|^{p_k}\right]^\beta$$

$$\begin{aligned}
&= \frac{1}{h_r^\alpha}\left[\sum_{\substack{k\in I_r\\ |\delta(z;A_k,B_k)-L|\geq\varepsilon}}|\delta(z;A_k,B_k)-L|^{p_k}\right]^\beta + \frac{1}{h_r^\alpha}\left[\sum_{\substack{k\in I_r\\ |\delta(z;A_k,B_k)-L|<\varepsilon}}|\delta(z;A_k,B_k)-L|^{p_k}\right]^\beta\\
&\leq \frac{1}{h_r^\alpha}\left[\sum_{\substack{k\in I_r\\ |\delta(z;A_k,B_k)-L|\geq\varepsilon}}\max\{Q^h,Q^H\}\right]^\beta + \left[\frac{1}{h_r^\alpha}\sum_{\substack{k\in I_r\\ |\delta(z;A_k,B_k)-L|<\varepsilon}}\max\{(\varepsilon)^{p_k}\}\right]^\beta\\
&\leq \max\{Q^h,Q^H\}.\frac{1}{h_r^\alpha}\Big[|\{k\in I_r: |\delta(z:A_k,B_k)-L|\geq\varepsilon\}|\Big]^\beta + \max\{\varepsilon^h,\varepsilon^H\}.
\end{aligned}$$

Therefore, $\{A_k\}_{(\alpha,\beta)} \overset{[WN]_\theta^{L(p)}}{\sim} \{B_k\}_{(\alpha,\beta)}$. Q.E.D.

Definition 2.5. Consider a metric space $(Z,\mathcal{J})$, $0<\alpha\leq\beta\leq 1$. For A_k, B_k be any closed non-empty subsets of Z, then the sequences $\{A_k\}$ and $\{B_k\}$ are strongly Cesàro asymptotically equivalent of multiplicity L of order (α,β) if for each $z\in Z$,

$$\lim_m \frac{1}{m^\alpha}\left[\sum_{k=1}^m|\delta(z;A_k,B_k)-L|\right]^\beta = 0$$

represented as $\{A_k\}_{(\alpha,\beta)} \overset{[W\sigma_1]}{\sim} \{B_k\}_{(\alpha,\beta)}$ are simply strongly Cesàro asymptotically equivalent of order (α,β) if $L=1$.

Definition 2.6. Consider a metric space $(Z,\mathcal{J})$, $0<\alpha\leq\beta\leq 1$ and a positive real numbers sequence $p=(p_k)$. For A_k, B_k be any closed non-empty subsets of Z, then $\{A_k\}$ and $\{B_k\}$ are strongly Cesàro asymptotically p-equivalent of multiplicity L of order(α,β) if for each $z\in Z$,

$$\lim_m \frac{1}{m^\alpha}\left[\sum_{k=1}^m|\delta(z;A_k,B_k)-L|^{p_k}\right]^\beta = 0$$

(represented as $\{A_k\}_{(\alpha,\beta)} \overset{[W\sigma_{(p)}]}{\sim} \{B_k\}_{(\alpha,\beta)}$) and simply strongly Cesàro asymptotically p-equivalent of order (α,β) if $L=1$.

Theorem 2.7. Consider a metric space $(Z,\mathcal{J})$, $0<\alpha\leq\beta\leq 1$ and $\theta=\{k_r\}$ be a lacunary sequence and A_k, B_k be non-empty subsets of Z with $\liminf_r q_r > 1$, then

$$\{A_k\}_{(\alpha,\beta)} \overset{[W\sigma_{(p)}]}{\sim} \{B_k\}_{(\alpha,\beta)} \text{ implies } \{A_k\}_{(\alpha,\beta)} \overset{[WN]_\theta^{L(p)}}{\sim} \{B_k\}_{(\alpha,\beta)}.$$

Proof. Since $\liminf_r q_r > 1$. Then there exists $\eta>0$ such that $q_r\geq 1+\eta$ for all $r\geq 1$. Thus, for $\{A_k\}_{(\alpha,\beta)} \overset{[W\sigma_{(p)}]}{\sim} \{B_k\}_{(\alpha,\beta)}$,we have

$$A_r = \frac{1}{h_r^\alpha}\left[\sum_{k\in I_r} |\delta(z;A_k,B_k)-L|^{p_k}\right]^\beta$$

$$= \frac{1}{h_r^\alpha}\left[\sum_{k=1}^{k_r} |\delta(z;A_k,B_k)-L|^{p_k}\right]^\beta - \frac{1}{h_r^\alpha}\left[\sum_{k=1}^{k_{r-1}} |\delta(z;A_k,B_k)-L|^{p_k}\right]^\beta$$

$$= \frac{K_r^\alpha}{h_r^\alpha}\left(\frac{1}{k_r^\alpha}\left[\sum_{k=1}^{k_r} |\delta(z;A_k,B_k)-L|^{p_k}\right]^\beta\right) - \frac{k_{r-1}^\alpha}{h_r^\alpha}\left(\frac{1}{k_{r-1}^\alpha}\left[\sum_{k=1}^{k_{r-1}} |\delta(z;A_k,B_k)-L|^{p_k}\right]^\beta\right)$$

Since $h_r^\alpha = k_r^\alpha - k_{r-1}^\alpha$, we have $\frac{k_r^\alpha}{h_r^\alpha} \leq \frac{1+\eta}{\eta}$ which leads to

$$\frac{1}{k_{r-1}^\alpha}\left[\sum_{k=1}^{k_{r-1}} |\delta(z;A_k,B_k)-L|^{p_k}\right]^\beta$$

and

$$\frac{1}{k_r^\alpha}\left[\sum_{k=1}^{k_r} |\delta(z;A_k,B_k)-L|^{p_k}\right]^\beta$$

converges to zero. Thus, $\{A_k\}_{(\alpha,\beta)} \overset{[WN]_\theta^{L(p)}}{\sim} \{B_k\}_{(\alpha,\beta)}$. Q.E.D.

Theorem 2.8. Consider a metric space $(Z, \mathcal{J})$, $0 < \alpha \leq \beta \leq 1$ and $\theta = \{k_r\}$ be a lacunary sequence and A_k, B_k be non-empty subsets of Z with $\limsup_r q_r > 1$, then

$$\{A_k\}_{(\alpha,\beta)} \overset{[WN]_\theta^{L(p)}}{\sim} \{B_k\}_{(\alpha,\beta)} \text{ implies } \{A_k\}_{(\alpha,\beta)} \overset{[W\sigma_{(p)}]}{\sim} \{B_k\}_{(\alpha,\beta)}$$

Proof. Since $\limsup_r q_r < \infty$, then there is $Q > 0$ such that $q_r < Q$ for all $r \geq 1$. Let $\{A_k\}_{(\alpha,\beta)} \overset{[WN]_\theta^{L(p)}}{\sim} \{B_k\}_{(\alpha,\beta)}$ and $\varepsilon > 0$, there exist $R > 0$ such that for every $j \geq R$ and

$$A_j = \frac{1}{h_j^\alpha}\left[\sum_{k\in I_j} |\delta(z;A_k,B_k)-L|^{p_k}\right]^\beta \leq \varepsilon.$$

we can also find $H > 0$ such that $A_j < H$ for all $j = 1, 2, ...$
Now let t be any integer with satisfying $k_{r-1} < t \leq k_r$, where $r > R$.
Then, we can write

$$\frac{1}{t^\alpha}\left[\sum_{k=1}^{t}|\delta(z;A_k,B_k)-L|^{p_k}\right]^\beta$$

$$\begin{aligned}
&\leq \frac{1}{k_{r-1}^\alpha}\left[\sum_{k=1}^{k_r}|\delta(z;A_k,B_k)-L|^{p_k}\right]^\beta \\
&= \frac{1}{k_{r-1}^\alpha}\left(\left[\sum_{k\in I_1}|\delta(z;A_k,B_k)-L|^{p_k}\right]^\beta+\left[\sum_{k\in I_2}|\delta(z;A_k,B_k)-L|^{p_k}\right]^\beta+\ldots+\left[\sum_{k\in I_r}|\delta(z;A_k,B_k)-L|^{p_k}\right]^\beta\right) \\
&= \frac{k_1}{k_{r-1}^\alpha}A_1+\frac{k_2-k_1}{k_{r-1}^\alpha}A_2+\ldots+\frac{k_R-K_{R-1}}{k_{r-1}^\alpha}A_R+\frac{k_{R+1}-k_R}{k_{r-1}^\alpha}A_{R+1}+\ldots+\frac{k+r-k_{r-1}}{k_{r-1}^\alpha}A_r \\
&\leq \{\sup_{j\geq 1}A_j\}\frac{k_R}{k_{r-1}^\alpha}+\{\sup_{j\geq 1}A_j\}\frac{k_r-k_R}{k_{r-1}^\alpha} \\
&\leq H.\frac{k_B}{k_{r-1}^\alpha}+\varepsilon Q.
\end{aligned}$$

This completes the proof. Q.E.D.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed equally to writing this paper. Both authors read and approved the manuscript.

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