Finite interpolation by distances in the disk

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Abstract

The aim of this note is to introduce a topic about some ways to interpolate values of a function, on a finite set of points in the disk. Each interpolating function is built by means of a certain distance between points. We study the behaviour of such interpolations.

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1 Introduction

Let f be a complex function on the open unit disk $\mathbb{D}$ of the complex plane and for $n \geq 2$, let $Z_n = \{z_1, \dots, z_n\}$ be a set of different points in $\mathbb{D}$. We recall that the interpolating polynomial of the values of the function f on Z_n, in the Lagrange form, is ([1]):

$$L[f](z) = \sum_{j=1}^{n} f(z_j) \prod_{\substack{i=1 \\ i \neq j}}^{n} \frac{z_i - z}{z_i - z_j}. \tag{1.1}$$

The fact that the error function

$$L^*[f](z) = |\, L[f](z) - f(z)\, |, \quad z \in \mathbb{D},$$

is bounded in terms of distances between points motivates us to pose the question: what happens if, looking for uniformity, we give up analyticity of the interpolating polynomial in exchange for also having an interpolating function in terms of distances?

For a given distance $d(w, z)$ in $\mathbb{D}$, we define similarly to (1.1) the interpolating function

$$d[f](z) = \sum_{j=1}^{n} f(z_j) \prod_{\substack{i=1 \\ i \neq j}}^{n} \frac{d(z_i, z)}{d(z_i, z_j)}.$$

Its interest is that to interpolate a set of values $w_j := f(z_j)$, $j = 1, \dots, n$, it is not necessary to use the points z_j, just the distance between them and their distance from the variable. We consider three distances: the Euclidean $E(w, z) = |z - w|$, the pseudo-hyperbolic

$$\rho(w, z) = \left| \frac{z - w}{1 - \overline{w} z} \right|$$

and the hyperbolic

$$\psi(w, z) = \log \frac{1 + \rho(w, z)}{1 - \rho(w, z)}.$$

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Our goal is to analyse the error function

$$d^*[f](z) = |\, d[f](z) - f(z)\,|, \quad z \in \mathbb{D},$$

for each of the distances above.

We point out that for a polynomial P it is possible that $d^*[P](z) \le L^*[P](z)$, and for a distance function D, that $L^*[D](z) \le d^*[D](z)$:

Example 1.1. Let $P(z) = z^2$ and $Z_2 = \{-0.4, 0.6\}$. We have:

$$\begin{aligned} E^*[P](z) &= 0.16\,|z - 0.6| + 0.36\,|z + 0.4|, \\ L^*[P](z) &= 0.2z + 0.24, \end{aligned}$$

and $E^*[P](z) \le L^*[P](z)$ for any $z \in (-1, 1)$.

Example 1.2. Let $Z_2 = \{-0.5, -0.1\}$. If $D(z) = \rho(z, \frac{1}{2})$, then $L^*[D](z) \le E^*[D](z)$ for all $z \in (-1, 1)$.

Example 1.3. For $Z_2 = \{-0.5, -0.1\}$, $D(z) = |z|$ and $z \in (-1, 1)$, it follows that:

$$\begin{aligned} &E^*[D](z) < L^*[D](z), \text{ if } z > 1/7, \\ &E^*[D](z) = L^*[D](z), \text{ if } z \in (z_1, z_2), \\ &E^*[D](z) > L^*[D](z), \text{ otherwise.} \end{aligned}$$

On the other hand, we note that all inequalities between $E^*[f]$, $\rho^*[f]$ and $\psi^*[f]$ happen. We write $d^*(z) := d^*[1](z)$.

Example 1.4. If $Z_2 = \{0.5, 0.8\}$, then:

$$\begin{aligned} &\rho^*(0.1) \approx 1.36 < \psi^*(0.1) \approx 1.63 < E^*(0.1) \approx 2.67 \\ &E^*(0.7) = \psi^*(0.7) = 0 < \rho^*(0.7) \approx 0.07 \\ &E^*(0.9) \approx 0.67 < \rho^*(0.9) \approx 1.17 < \psi^*(0.9) \approx 1.36 \end{aligned}$$

Definition 1.1. We say that a function f is exact for the polynomial interpolation (resp. for the distance interpolation) if $L^*[f](z) = 0$ (resp. $d^*[f](z) = 0$) for all z in $\mathbb{D}$.

It is well-known that any polynomial p_{n-1} of degree at most $n-1$ is exact for the polynomial interpolation. On the other hand, it is an easy computation to show that the function $\hat{d}$ defined in $\mathbb{D}$ by

$$\hat{d}(z) = k \sum_{j=1}^{n} \prod_{\substack{i=1 \\ i \neq j}}^{n} d(z_i, z),$$

where from now on k will denote a complex constant different from zero, is exact for the distance interpolation. We are interested in knowing how the exact functions for one interpolation behave with respect to the other. More specifically, we want to estimate: $L^*[\hat{d}](z)$ and, specially, $d^*[p_{n-1}](z)$.

2 Statement of results

First, we confine ourselves to Z_2. For any three points z, η, w in $\mathbb{D}$, we put

$$d(z,\eta,w) := d(z,\eta) + d(\eta,w) - d(z,w).$$

It is easy to get

$$L^*[\hat{d}](z) = k\, d(z_1, z, z_2).$$

We have $L^*[\hat{E}] < 4k$, since $|z - z_1|$, $|z - z_2| < 2$, and $L^*[\hat{\rho}] < 2k$, because $\rho(z, z_1)$, $\rho(z, z_2) < 1$. In the particular case that Z_2 is on a diameter Δ of $\mathbb{D}$, then $L^*[\hat{E}] < 2k$ and

$$\lim_{|z_1 - z_2| \to 1} \sup_{z \in \Delta} L^*[\hat{\rho}](z) = k.$$

We will estimate $d^*[p_1](z)$ for each distance and the test polynomials: $p_1(z) \equiv k$ and $p_1(z) = z - (z_1 + z_2)/2$.

For the pseudo-hyperbolic distance, we will use the following improvement of the triangle inequality ([2]):

Lemma 2.1. For any three points z, η, w in $\mathbb{D}$,

$$\rho(z,\eta,w) \geq \rho(z,w)\rho(z,\eta)\rho(\eta,w).$$

We search for upper bounds for $d^*[k](z)$, with less possible dependence on z, and taking into account that $d^*[k](z) = |k|\, d^*(z)$, we may assume without loss of generality $k = 1$.

Proposition 2.1. For all $z \in \mathbb{D}$, the following bounds hold:

$$E^*(z) < \frac{4}{|z_1 - z_2|} \tag{2.1}$$

$$\rho(z_1, z)\, \rho(z_2, z) \leq \rho^*(z) < \frac{2}{\rho(z_1, z_2)} \tag{2.2}$$

$$\psi^*(z) < \frac{2}{\rho(z_1, z_2)\, [1 - \rho(z_1, z)]\, [1 - \rho(z_2, z)]} \tag{2.3}$$

Proof. By the triangle inequality,

$$E^*(z) = \frac{E(z_1, z, z_2)}{|z_1 - z_2|}, \tag{2.4}$$

and using $|z - z_1|, |z - z_2| < 2$, it follows (2.1). By the triangle inequality,

$$\rho^*(z) = \frac{\rho(z_1, z, z_2)}{\rho(z_1, z_2)} \tag{2.5}$$

and using $\rho(z, z_1), \rho(z, z_2) < 1$, the upper bound in (2.2) follows. The lower bound in (2.2) is immediate from Lemma 2.1. By the triangle inequality,

$$\psi^*(z) = \frac{\psi(z_1, z, z_2)}{\psi(z_1, z_2)} \tag{2.6}$$

Since for $t \geq 0$,

$$2t \leq \log \frac{1+t}{1-t} \leq \frac{2t}{1-t}, \tag{2.7}$$

then

$$\psi^*(z) < \frac{\dfrac{\rho(z_1, z)}{1-\rho(z_1, z)} + \dfrac{\rho(z_2, z)}{1-\rho(z_2, z)}}{\rho(z_1, z_2)}, \tag{2.8}$$

and using $\rho(z, z_1), \rho(z, z_2) < 1$, it follows (2.3). Q.E.D.

Note that $E^*(z) = \psi^*(z) = 0$ for any z in the segment $[z_1, z_2]$, and $\rho^*(z) \neq 0$ for all z different from z_1 and z_2.

Proposition 2.2. For any $z \in \mathbb{D}$, the following relationships between $E^*(z)$, $\rho^*(z)$ and $\psi^*(z)$ hold:

$$E^*(z) \leq \frac{2\,[\rho^*(z)+1]}{1-|z_1|\,|z_2|} \tag{2.9}$$

$$\rho^*(z) \leq \frac{2\,[E^*(z)+1]}{1-|z|\ \max\{|z_1|, |z_2|\}} \tag{2.10}$$

$$\psi^*(z) \leq \frac{\rho^*(z)+1}{1-\max\{\rho(z_1, z), \rho(z_2, z)\}} \tag{2.11}$$

Proof. Using that

$$\frac{|z-w|}{2} \leq \rho(z, w) \leq \frac{|z-w|}{1-|z|\,|w|},$$

we have from (2.4) and (2.5):

$$E^*(z) \leq \frac{2\rho^*(z)+1+|z_1|\,|z_2|}{1-|z_1|\,|z_2|}$$
$$\rho^*(z) \leq \frac{2E^*(z)+1+|z|\ \max\{|z_1|, |z_2|\}}{1-|z|\ \max\{|z_1|, |z_2|\}},$$

and (2.9) and (2.10) follow. Finally, (2.11) is a consequence of (2.8). Q.E.D.

Proposition 2.3. Let $q(z)$ be the polynomial $q(z) = z - (z_1 + z_2)/2$. For all $z \in \mathbb{D}$, the following bounds hold:

$$E^*[q](z) < 4$$

$$\rho^*[q](z) < 4$$

$$\psi^*[q](z) \leq 2 + \frac{1}{\rho(z_1, z_2)} \left[\frac{1}{1-\rho(z_1, z)} + \frac{1}{1-\rho(z_2, z)}\right] \tag{2.12}$$

Proof. By the triangle inequality,

$$E^*[q](z) \le |z-z_1| + |z-z_2| < 4$$

and

$$\rho^*[q](z) \le \frac{|1-\overline{z_1}z_2|\,[\rho(z_1,z)+\rho(z_2,z)] + |z-z_1| + |z-z_2|}{2} < 4.$$

By the triangle inequality and (2.7),

$$\begin{aligned}\psi^*[q](z) \le & \frac{1}{2\rho(z_1,z_2)}\left(\frac{|z-z_1|\,\rho(z_1,z)}{1-\rho(z_1,z)} + \frac{|z-z_2|\,\rho(z_2,z)}{1-\rho(z_2,z)}\right) \\ & + \frac{|z-z_1|+|z-z_2|}{2}\end{aligned}$$

and the bound in (2.12) follows. Q.E.D.

Next, we provide some bounds for n points ($n \ge 2$). We write $E_n^*(z)$ instead $E^*(z)$, put

$$F_n(z) := \sum_{j=1}^{n} \prod_{\substack{i=1\\ i\neq j}}^{n} \frac{|z_i - z|}{|z_i - z_j|}$$

and consider two particular cases:

1. Z_n **is on a diameter of** $\mathbb{D}$ **and** $|z_i - z_{i+1}| = |z_{i+1} - z_{i+2}|$**,** $i = 1, \ldots, n-2$**.**

Since $|z_i - z_{i+1}| < \frac{2}{n-1}$ and $|z - z_j| > 1$, $j = 1, \ldots, n$, then for any $z \in \mathbb{D}$:

$$E_n^*(z) = F_n(z) - 1 > n\left(\frac{n-1}{2}\right)^{n-1} - 1.$$

For some values of n:

$$E_3^*(z) > 2, \quad E_4^*(z) > \frac{25}{2}, \quad E_5^*(z) > 79, \quad E_6^*(z) > \frac{9359}{16} \approx 584,94.$$

On the other hand, $|z - z_j| < \sqrt{2}$, $j = 1, \ldots, n$, and if we denote $s := |z_i - z_{i+1}|$, then for all $z \in \mathbb{D}$:

$$E_n^*(z) < F_n(z) < \left(\frac{\sqrt{2}}{s}\right)^{n-1} \sum_{j=1}^{n} \prod_{\substack{i=1\\ i\neq j}}^{n} \frac{1}{|i-j|}.$$

For some values of n:

$$E_2^*(z) < \frac{2\sqrt{2}}{s}, \quad E_3^*(z) < \frac{4}{s^2}, \quad E_4^*(z) < \frac{8\sqrt{2}}{3s^3},$$

$$E_5^*(z) < \frac{8}{3s^4}, \quad E_6^*(z) < \frac{16\sqrt{2}}{15s^5}.$$

2. Z_n **(**$n \ge 3$**) is a regular polygon with respect to the Euclidean distance.**

The minimum of the function $F_n(z)$ is reached at the centre of the polygon, $F_n(z) \geq 1$ and, particularly, $F_3(z) \geq 4/3$. Then, for any $z \in \mathbb{D}$:

$$E_n^*(z) < F_n(z) < n2^{n-1} \prod_{j=2}^{n} \frac{1}{|z_1 - z_j|}$$

and also, $E_3^*(z) \geq 1/3$. If l denotes the side of the polygon, then for some values of n:

$$E_3^*(z) < \frac{12}{l^2}, \quad E_4^*(z) < \frac{16\sqrt{2}}{l^3} \approx \frac{22.63}{l^3},$$

$$E_5^*(z) < \frac{80 \sin^2(\pi/5)}{l^4 \sin^2(3\pi/5)} \approx \frac{30.56}{l^4}, \quad E_6^*(z) < \frac{32}{l^5}.$$

If the polygon is on the boundary of $\mathbb{D}$, then the maximum of $F_n(z)$ is reached at the middle point of two consecutive vertices. In that case, for some values of n:

$$E_2^*(z) < \sqrt{2}, \quad E_3^*(z) < \frac{5}{3}, \quad E_4^*(z) < \sqrt{2+\sqrt{2}} + \sqrt{2-\sqrt{2}} \approx 2.61.$$

Also, $\rho^*(z) = n - 1$.

Remark 2.1. We point out that it is possible to introduce a kind of hybrid interpolation: polynomial interpolation and distance interpolation or/and this last with different distances at a time. For example, for $n \geq 4$, $0 \leq n_1, n_2, n_3, n_4 \leq n-3$ and $n_1 + n_2 + n_3 + n_4 = n$, we can define (reordering the points, if necessary):

$$L_{n_1} E_{n_2} \rho_{n_2} \psi_{n_4}[f](z) = \sum_{j=1}^{n_1} f(z_j) \prod_{\substack{i=1 \\ i \neq j}}^{n_1} \frac{z_i - z}{z_i - z_j}$$

$$+ \sum_{j=n_1+1}^{n_1+n_2} f(z_j) \prod_{\substack{i=n_1+1 \\ i \neq j}}^{n_1+n_2} \frac{|z_i - z|}{|z_i - z_j|} + \sum_{j=n_1+n_2+1}^{n_1+n_2+n_3} f(z_j) \prod_{\substack{i=n_1+n_2+1 \\ i \neq j}}^{n_1+n_2+n_3} \frac{\rho(z_i, z)}{\rho(z_i, z_j)}$$

$$+ \sum_{j=n_1+n_2+n_3+1}^{n} f(z_j) \prod_{\substack{i=n_1+n_2+n_3+1 \\ i \neq j}}^{n} \frac{\psi(z_i, z)}{\psi(z_i, z_j)}.$$

Remark 2.2. In this context, we can also consider a type of weak interpolation, in the sense of looking for an interpolating polynomial Q such that $|Q| = |f|$ on Z_n. Clearly, $L[\,|f|\,]$ works, but if we know the Euclidean distances between $z_1, \ldots, z_n$, then it is more convenient to use the mixed function (contains distances) $M[f]$ defined by

$$M[f](z) = \sum_{j=1}^{n} |f(z_j)| \prod_{\substack{i=1 \\ i \neq j}}^{n} \frac{z_i - z}{|z_i - z_j|}.$$

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