Towards solving linear fractional differential equations with Hermite operational matrix

Hatice Yalman Kosunalp[1] and Mustafa Gulsu[2]

[1]Bandırma Onyedi Eylul University, Gonen Vocational School, Department of Accounting and Tax, Gonen, Balikesir, Türkiye
[2]Mugla Sitki Kocman University, Faculty of Science, Mathematics Department, Mugla, Türkiye
E-mail: hkosunalp@bandirma.edu.tr[1], mgulsu@mu.edu.tr[2]

Abstract

This paper presents the derivation of a new operational matrix of Caputo fractional derivatives through Hermite polynomials with Tau method to solve a set of fractional differential equations (FDEs). The proposed algorithm is intended to solve linear type of FDEs with the pre-defined conditions into a matrix form for redefining the complete problem as a system of a algebraic equations.The proposed strategy is then applied to solve the simplified FDEs in linear form. To assess the performance of the proposed method, exact and approximate solutions for a number of illustrative examples are obtained which prove the effectiveness of the idea.

2010 Mathematics Subject Classification. **44A45**. 26A33.
Keywords. fractional differential equations, hermite, operational matrix, caputo.

1 Introduction

Fractional calculus has been utilized in various scientific domains through developing efficient models with mathematical tools. Typical applications examples are electrical engineering, fluid dynamics [1], physics [2, 3, 4], economics [5], natural sciences [6], chemistry [7] and so on. As a recent study, the behavior of control processes in many practical systems has been identified with fractional calculus [8]. In order to describe these models, a significant emphasis is placed on the solutions of fractional differential equations (FDEs). A lot of numerical and analytical methods have been conducted to provide either approximate or exact solutions [9, 10, 11, 12, 13].

Spectral methods have been shown to be very efficient and accurate in solving different types of differential equations. This attracted many researchers to exploit them for the solutions of the FDEs. To solve linear multi-term FDEs, an operational matrix (OM) for chebyshev polynomials was derived with the sense of Caputo fractional derivative with initial conditions [14]. Authors in [15] presented a similar OM of shifted-Legendre polynomials to be applied with spectral approaches to solve linear and nonlinear FDEs. Another work extracts the OM for fractional integration in Genocchi polynomials [16]. Two previous similar works attempted to derive the OMs for the polynomials of Jacobi [17], Legendre [18] and Boubaker [19]. Although these studies presented a similar structure with numerical results, there are still some drawbacks. A critical issue is the complexity associated with the number of iterations in solving the FDEs. Some of the studies obtain good results with high number of iterations which may mitigate the speed of the solution. On the other hand, some polynomials such as Laugerre introduce extra coefficients which can also increase the complexity of the solution. The number of examples solved is relatively low in some studies to support the actual performance in which only exact solutions in simple form of polynomials are considered.

Advanced Studies: Euro-Tbilisi Mathematical Journal 16(2) (2023), pp. 47–61.
DOI: 10.32513/asetmj/193220082316
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 24 March 2023.
Accepted for publication: 28 April 2023.

This work explores the potential to extend the current status of state-of-the-art studies, in order to provide a fast,accurate and efficient way. A distinctive feature of this work is its lightweight mechanism for satisfying the complexity requirements, where high complex solutions cannot be implemented practically in real-world problems. To achieve the entire goal, we introduce Hermite polynomials with operational matrix strategy to be an effective solution in the relevant field. The main motivation of this paper is to compute Hermite OM of Caputo fractional derivative in association with spectral tau method for solving linear FDEs, subject to initial equations. The underlying property of the proposed idea is to convert the FDEs system into algebraic forms which is eventually given in matrix sense. Therefore, the complete problem actually requires the solution of algebraic equations. The performance of the proposed strategy is tested through illustrative examples achieving high accurate outputs. The other parts of the paper are organised in following way. Section 2 presents the required preliminaries and notations about Caputo derivatives and a brief introduction to the properties of Hermite polynomials. The Hermite OM for fractional derivative is computed in Section 3. This matrix is then applied together with the method of spectral tau for the solution of linear FDEs in Section 4. Section 5 presents a number of representative examples with the implementation of the proposed solution. The paper is concluded in Section 6.

2 Preliminaries and notation

We start with reviewing a number of the required definitions of fractional derivatives and Hermite polynomials.

Definition 2.1. The Rieman-Liouville fractional integral for order v is given by

$$J^v f(x) = \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} f(t)dt, J^0 f(x) = f(x). \tag{2.1}$$

$$J^0 f(x) = f(x). \tag{2.2}$$

Definition 2.2. The Caputo derivative operator with order v satisfying the condition of $v > 0$ is defined as

$$D^v f(x) = J^{m-v} D^m = \frac{1}{\Gamma(m-v)} \int_0^x (x-t)^{(m-v-1)} \frac{d^m}{dt^m} f(t)dt, m-1 < v < m, x > 0. \tag{2.3}$$

and for the exponential function this equation also holds true:

$$D^v x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-v)} x^{\beta-v} \tag{2.4}$$

$$D^v x^\beta == \begin{cases} 0, & \text{for } \beta \in N and \beta < \lceil v \rceil \\ \frac{\Gamma(\beta+1)}{\Gamma \beta+1-v} x^{\beta-v}, & \text{for } \beta \in N \text{ and } \beta \geq \lfloor v \rfloor \end{cases}$$

where $\lfloor v \rfloor$ and $\lceil v \rceil$ represents the functions of floor and ceiling in turn, assuming N is the sequential numbers starting from 1, $N = 1, 2, 3, ...$

The Caputo derivation operator is linear such as

$$D^v(\lambda f(x) + \mu g(x)) = \lambda D^v f(x) + \mu D^v g(x), \tag{2.5}$$

where λ and μ take constant numbers.

We note that when $v \in N$ the definition of the Caputo derivative and usual differential operator definition coincides.

2.1 Properties of Hermite Polynomials

The Hermite polynomials are defined on $(-\infty, \infty)$ with this analytic formula [20]:

$$H_i(x) = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \frac{(-1)^k i!(2x)^{i-2k}}{k!(i-2k)!} \tag{2.6}$$

or it is defined as

$$H_i(x) = (-1)^i e^{-x^2} \frac{d^i e^{-x^2}}{dx^i} \tag{2.7}$$

Hermite polynomials are orthogonal polynomials so they satisfy this equation:

$$\int_{-\infty}^{\infty} H_i(x)H_j(x) = \begin{cases} 0, & i \neq j \\ 2^i i!\sqrt{\pi}, & i = j \end{cases} \tag{2.8}$$

where e^{-x^2} is the weight function of Hermite polynomials. Hermite polynomials satisfy this recurrence relation:

$$H_{i+1}(x) = 2xH_i(x) - 2iH_{i+1}(x) \tag{2.9}$$

2.2 The Caputo fractional derivatives of $H_i(x)$

A function $u(x)$ which is square integrable in $(-\infty, \infty)$ can be defined by Hermite polynomials like

$$u(x) = \sum_{j=0}^{\infty} c_j H_j(x) \tag{2.10}$$

then the coefficients c_j is defined as

$$c_j = \frac{1}{2^j j!\sqrt{\pi}} \int_{-\infty}^{\infty} u(x)H_j(x)w(x)dx, j = 0, 1, 2, ... \tag{2.11}$$

If we consider the truncated $N + 1$ Hermite polynomials, it can be written

$$u(x) = \sum_{j=0}^{N} c_j H_j(x) = C^T \varphi(x) \tag{2.12}$$

where the C is the unknown coefficient vector and $\varphi(x)$ is the Hermite vector as illustrated below

$$C^T = [c_0, c_1, ..., c_N] \tag{2.13}$$

$$\varphi(x) = [H_0(x), H_1(x), ..., H_N(x)] \tag{2.14}$$

The first derivation of vector $\varphi(x)$ is given as

$$\frac{d\varphi(x)}{dx} = D^1\varphi(x) \tag{2.15}$$

where $D^{(1)}$ indicates an OM with the size of $(N + 1) \times (N + 1)$. Then the powers of D^1 for $n \in N$ is given as

$$D^{1^{(n)}}\varphi(x) = (D^{(1)})^{(n)}\varphi(x) \tag{2.16}$$

Theorem 2.1 Let $\varphi_i(x)$ is the Hermite polynomial and if $v \geq 0$; then

$$D^v \varphi(x) = D^{(v)} \varphi(x) \tag{2.17}$$

where $D^{(v)}$ represents OM of Hermite polynomials of $(N+1) \times (N+1)$ length which is defined as

$$\mathbf{D}^v = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & 0 \\ \Omega_v(i,0) & \Omega_v(i,1) & \Omega_v(i,2) & \dots & \Omega_v(i,N) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \Omega_v(N,0) & \Omega_v(N,1) & \Omega_v(N,2) & \dots & \Omega_v(N,N) \end{pmatrix}$$

where

$$\Omega_v(i,j) = \sum_{k=0}^{\lfloor \frac{i-\lfloor v \rfloor}{2} \rfloor} \frac{1}{2^j j! \sqrt{\pi}} \sum_{r=0}^{\lfloor j/2 \rfloor} \frac{(-1)^(k+r) 2^{(i-2k+j-2r)} i! j! \frac{\Gamma(i-2k+j-2r+1)}{2}}{(j-2r)! k! r! \Gamma(i-2k+1-v)}, j = 0, 1, ..., N \tag{2.18}$$

`Proof.` We will start proof with the analytic formula of Hermite Polynomials such as

$$H_i(x) = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \frac{(-1)^k (2x)^{i-2k}}{k!(i-2k)!} \tag{2.19}$$

it is equal to:

$$H_i(x) = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \frac{(-1)^k 2^{i-2k} x^{i-2k}}{k!(i-2k)!} \tag{2.20}$$

By approximating x^{i-2k} function with Hermite polynomials, we obtain

$$x^{i-2k} = \sum_{j=0}^{N} c_j H_j(x) \tag{2.21}$$

then c_j can be found like this:

$$c_j = \frac{1}{2^j j! \sqrt{\pi}} \int_{-\infty}^{\infty} H_j(x) x^{i-2k} e^{-x^2} dx \tag{2.22}$$

If we obtain Caputo derivative of x^{i-2k} from the analytic formula, it is:

$$D^v H_i(x) = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \frac{(-1)^k 2^{i-2k} D^v (x^{i-2k})}{k!(i-2k)!} \tag{2.23}$$

From the definiton of the Caputo derivative of exponential function we get the derivative of x^{i-2k}

$$D^v x^{i-2k} = \frac{\Gamma(i-2k+1)}{\Gamma(i-2k-v+1)} x^{i-2k-v}, i = \lceil v \rceil, \lceil v \rceil + 1, ..., N. \tag{2.24}$$

If we substitute this definition into the Eq.(2.22) it will be

$$D^v H_i(x) = \sum_{k=0}^{\lfloor \frac{i-\lceil v \rceil}{2} \rfloor} \frac{(-1)^k 2^{i-2k} \frac{\Gamma(i-2k-v)}{\Gamma(i-2k-v+1)} x^{i-2k-v}}{k!(i-2k)!}, i = \lceil v \rceil, \lceil v \rceil + 1, ..., N. \tag{2.25}$$

If we employ Eqs.(2.23) -(2.25) we obtain

$$D^v H_i(x) = \sum_{j=0}^{N} \Omega_v(i,j) H_j(x) \tag{2.26}$$

When extracting $\Omega_v(i,j)$ from Eq.(2.25), it is:

$$\Omega_v(i,j) = \sum_{k=0}^{\lfloor \frac{i-\lfloor v \rfloor}{2} \rfloor} \frac{1}{2^j j! \sqrt{\pi}} \sum_{r=0}^{\lfloor j/2 \rfloor} \frac{(-1)^(k+r) 2^{(i-2k+j-2r)} i! j! \frac{\Gamma(i-2k+j-2r+1)}{2}}{2(j-2r)! k! r! \Gamma(i-2k+1-v)}, j = 0, 1, ..., N \tag{2.27}$$

3 Applications of operational matrix for linear FDEs

In this part of the paper, we use Tau method after composing OM of Hermite polynomials for the solution of FDEs. Here, linear FDEs are only considered, that is given as

$$D^v u(x) = \sum_{i=1}^{k} \gamma_i D^{\beta_j} u(x) + \gamma_{k+1} u(x) + g(x) \tag{3.1}$$

which is combined with the initial conditions given below

$$u^{(i)}(0) = d_i, i = 0, 1, ..., m-1. \tag{3.2}$$

where i takes real consecutive constants ranging from 0 to 1 , ... , k, $m-1 < v < m$ and $0 < \beta_1 < \beta_2 < ... < \beta_k < v$ and $g(x)$ is the source function given. Also, $D^v u(x) = u^{(v)}(x)$ is the fractional derivative with order v for $u(x)$ in Caputo sense and the d_i values $d_i, i = (0, 1, ..., m-1)$ is the initial conditions.

For solving the initial value problem (3.1)-(3.2) approximating $u(x)$ and $g(x)$ by Hermite polynomials as

$$u(x) = \sum_{i=0}^{N} c_i H_i(x) = C^T \varphi(x) \tag{3.3}$$

and

$$g(x) = \sum_{i=0}^{N} g_i H_i(x) = G^T \varphi(x) \tag{3.4}$$

vector $G = [g_0, g_1, ..., g_N]$ is known but $C = [c_0, c_1, ..., c_N]$ is unknown, that is what we aim to find. By using Theorem 2.1 and the relation (16) and (29) we obtain

$$D^v u(x) \simeq C^T D^v \varphi(x) \simeq C^T D^{(v)} \varphi(x) \tag{3.5}$$

$$D^{\beta_j} u(x) \simeq C^T D^{\beta_j} \varphi(x) \simeq C^T D^{(\beta_j)} \varphi(x). \tag{3.6}$$

Applying (30)-(33) the residual is

$$R_N(x) = \left(C^T \mathbf{D}^{(v)} - C^T \sum_{j=1}^{k} \gamma_j \mathbf{D}^{\beta_j} - \gamma_{k+1} C^T - G^T \right) \varphi(x) \tag{3.7}$$

Applying Tau method we get

$$\langle R_N(x), H_j(x) \rangle = \int_{-\infty}^{\infty} R_N(x) H_j(x) dx = 0 \tag{3.8}$$

and by substituting (11) and (30) into (29) for the conditions we obtain

$$u^{(i)}(0) = C^T \mathbf{D}^{(i)}(0) = d_i, i = 0, 1, ..., m-1. \tag{3.9}$$

With Eq. (3.8) we get $N - m + 1$ and with Eq. (3.9) we get m equations. By combining these two sets of algebraic equations we have $N + 1$ linear equtions and then we solve this linear FDE. The extra details can be found in [14].

4 Convergence analysis

Let $u(x) \in L^2(\Omega)$ be a function in a Hilbert space, $u_n(x) = \sum_{i=0}^{n} c_i H_i(x)$ be the best approximation polynomials of $u(x)$, then $u_n(x)$ converges to $u(x)$.

Let $u_n(x)$ be the approximate solution with a series of Hermite polynomials like

$$u(x) \simeq u_n(x) = \sum_{i=0}^{n} c_i H_i(x). \tag{4.1}$$

$u(x)$ is the exact solution and c_i is obtained as

$$c_i = \frac{1}{2^i i! \sqrt{\pi}} \int_\Omega u(x) H_i(x) w(x) dx \tag{4.2}$$

where $w(x)$ is the weight function of Hermite polynomials which is $w(x) = e^{-x^2}$.

Now let

$$\alpha_i =< u(x), H_i(x) >$$

Now we define partial sums of $\alpha_i H_i(x)$ as S_m. For $m \geq n$, S_m and S_n are an arbitrary sequence. If we prove that S_m is convergent in the Hilbert space, then $u_n(x)$ is proved to be convergent. S_m is defined as

$$S_m = \sum_{i=1}^{m} \alpha_i H_i(x)$$

Then in order to prove S_m as a Cauchy sequence in the Hilbert space, we consider the L^2-norm as:

$$\| S_m - S_n \|_L^2 = \| \sum_{i=n+1}^{m} \alpha_i H_i \|_L^2 \tag{4.3}$$

Then it is

$$\begin{aligned}
\| S_m - S_n \|_L^2 &= < \sum_{i=n+1}^{m} \alpha_i H_i, \sum_{j=n+1}^{n} \alpha_j H_j > && (4.4)\\
&= \int_\Omega w(x) \left(\sum_{i=n+1}^{m} \alpha_i H_i(x) \right) \left(\sum_{j=n+1}^{m} \alpha_j H_j(x) \right) dx && (4.5)\\
&= \sum_{i=n+1}^{m} \sum_{j=n+1}^{m} \alpha_i \alpha_j \int_\Omega w(x) H_i(x) H_j(x) dx && (4.6)
\end{aligned}$$

Then since

$$\int_\Omega w(x) H_i(x) H_j(x) dx = \delta_{ij}$$

From orthogonality we have

$$\delta_{ij} = \begin{matrix} 0 & i \neq j \\ 1 & i = j \end{matrix}$$

Then Eq.(4.6) will be

$$< \sum_{i=n+1}^{m} \alpha_i H_i, \sum_{j=n+1}^{n} \alpha_j H_j >= \sum_{i=n+1}^{m} {\alpha_i}^2 \tag{4.7}$$

From Bessel inequality in Eq.4.7 the series

$$\sum_{i=n+1}^{m} {\alpha_i}^2$$

is convergent series and hence $\| S_m - S_n \|^2 \to 0$ as $m, n \to \infty$

Then the proof is complete that is S_m is the Cauchy sequence.
`Theorem 4.1` (Bessel's Inequality)

$$\Phi_0(x), \Phi_1(x), \Phi_2(x), \ldots \Phi_n(x)$$

Let L_w^2 be orthogonal polynomials in a Hilbert space and let $u(x) \in L_w^2$ then $\sum_i \alpha_i \Phi(x)$ with weighing function $w(x)$ it must be true that

$$\int \left[u(x) - \sum_i \alpha_i \Phi(x) \right]^2 w(x) dx \geq 0$$

Proof.

$$\int \left[u(x) - \sum_i \alpha_i \Phi(x) \right]^2 w(x)dx = \int u^2(x)w(x)dx - 2\alpha_i \sum_i \int u(x)\Phi(x)w(x)dx - + \sum_i (\alpha_i)^2 \int \Phi^2(x)w(x)dx \geq 0 \tag{4.8}$$

The coefficients α_i can be found as

$$\alpha_i = \int u(x)\Phi(x)w(x)dx$$

Q.E.D.

then by considering
$2\alpha_i \sum_i \int u(x)\Phi(x)w(x)dx = 2\alpha_i^2$ and since $\int \Phi^2(x)w(x)dx = 1$ then

$$\sum_i (\alpha_i)^2 \int \Phi^2(x)w(x)dx = \sum_i (\alpha_i)^2$$

Then (4.8) will be

$$\int \left[u(x) - \sum_i \alpha_i \Phi(x) \right]^2 w(x)dx = \int u^2(x)w(x)dx - 2\sum_i \alpha_i^2 + \sum_i \alpha_i^2 \geq 0 \tag{4.9}$$

Then it is obvious that

$$\int u^2(x)w(x)dx \geq \sum_i \alpha_i^2 \tag{4.10}$$

It means that $\sum_i \alpha_i$ series is convergent. It is the direct result of Bessel's Inequality.

5 Numerical examples

Example 1. Our first example is non-homogeneous Bagley-Torvik equation initial value problem [17]

$$D^2u(x) + D^{3/2}u(x) + u(x) = 1 + x, \qquad u(0) = 0, u'(0) = 1 \tag{5.1}$$

This problem is solved exactly with

$$u(x) = 1 + x.$$

If we solve the problem for $N = 2$ by applying the technique we describe $u(x)$ and $g(x)$ will be

$$u(x) = \sum_{i=0}^{2} c_i H_i(x)$$

$$g(x) = \sum_{i=0}^{2} g_i H_i(x)$$

and we have

$$\mathbf{D}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 8 & 0 & 0 \end{pmatrix} \qquad \mathbf{D}^{(3/2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3.1205 & 2.3081 & 0.3901 \end{pmatrix}$$

and

$$G = \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix}$$

By using Eq.(3.8) we obtain

$$c_0 + 11.1205c_2 - 1 = 0 \tag{5.2}$$

and by appyling (Eq.3.9) we obtain

$$\begin{gathered} c_0 + 2c_2 - 1 = 0 \\ 2c_1 - 1 = 0 \end{gathered} \tag{5.3}$$

If we solve this linear system we get $c_0 = 1$, $c_1 = \frac{1}{2}$, $c_2 = 0$
and from here the solution we propose is

$$u(x) = C^T \varphi(x) = 1 + x.$$

that is the expected exact solution.

Example 2. Our next example is [21]

$$D^2u(x) + D^{3/2}u(x) + u(x) = x^2 + 2 + \frac{\Gamma(3)}{\Gamma(3/2)}x^{0.5}, \qquad u(0) = 0, u'(0) = 0 \tag{5.4}$$

Here, the solution is an exact one

$$u(x) = x^2$$

Here for $N = 2$ we have

$$\mathbf{D}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 8 & 0 & 0 \end{pmatrix} \qquad \mathbf{D}^{(3/2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3.1205 & 2.3081 & 0.3901 \end{pmatrix}$$

and

$$G = \begin{pmatrix} 3.2801 \\ 0.5770 \\ 4.0225 \end{pmatrix}$$

By applying technique described in (3.8)-(3.9) we get a c_0 value of 0.5, c_1 value of 0, c_2 value of 0.25
and from here the solution is

$$u(x) = x^2$$

the exact solution is achieved.
Example 3. Our third example is [22]

$$D^2u(x) - 2Du(x) + D^{1/2}u(x) + u(x) = x^3 - 6x^2 + 6x + \frac{16}{5\Gamma(1/2)}x^{2.5}$$

,

$$u(0) = 0, u'(0) = 0 \tag{5.5}$$

This problem has an exact solution of

$$u(x) = x^3$$

In this example for $N = 3$ we have the system like

$$c_0 - 3.2219c_1 + 9.5338c_2 - 0.9361c_3 - -2.5319 = 0$$

$$c_0 + 11.1205c_2 - 1 = 0 \tag{5.6}$$

From the conditions we have

$$c_0 - 2c_2 = 0$$

$$2c_1 - 12c_3 = 0$$

So we obtain C^T as

$$C^T = \begin{pmatrix} 0 \\ 0.75 \\ 0 \\ 0.125 \end{pmatrix}$$

The solution is extracted as

$$u(x) = C^T.\varphi(x) = x^3.$$

Example 4. Our next example is inhomogeneous linear equation [23]

$$D^\lambda u(x) + u(x) = \frac{2x^{2-\lambda}}{\Gamma(3-\lambda)} - \frac{x^{1-\lambda}}{\Gamma(2-\lambda)} + x^2 - x$$

$$u(0) = 0, 0 < \lambda \leq 1. \tag{5.7}$$

The solution is given in exact form as $u(x) = x^2 - x$.
We selected a λ value of 0.3

$$\mathbf{D}^{(0.3)} = \begin{pmatrix} 0,5533 & 0,2418 & -0,0415 \\ 0,6908 & 0,5533 & 0,1209 \\ 0,1953 & 0,8980 & 0,6363 \end{pmatrix}$$

By applying our technique to obtain a set of algebraic equations

$$c_0 + 0,78c_1 + 1,5388c_2 = 0,4946$$

$$1,577c_1 + 1,56c_2 = -0,3985$$

$$c_0 - 2c_2 = 0$$

Upon solution of the algebraic equations system with $N = 2$ we generate a c_0 value of 0.5, c_1 value of -0.5, c_2 value of 0.25.

Then, the solution is exactly found

$$u(x) = C^T.\varphi(x) = x^2 - x.$$

Example 5. The next example is given below [24]

$$D^2u(x) - 2Du(x) + D^{1/2}u(x) + u(x) = \beta^{1/2}e^{\beta x} + x^{2.5} + 3e^{\beta x}$$

$$u(0) = 1, u'(0) = \beta \tag{5.8}$$

The exact solution is given as $u(x) = e^{\beta x}$.

We solve this equation for different N and β. The range of β was chosen between 0.05 and 0.2 in its original paper. We extend this range from 0.2 to 0.5 to obtain more results. Also, the range of N is started by 2 until 4 as the error does not change significantly beyond this value. The results indicate a highly-accurate approximation towards the exact solution, which are given in table below. The results indicate that the error increases with varying the β and X values. This is because the exact solution is dependent on the values of β and X in which any increment on one of these values will increase the value of the exact solution. Therefore, the error between the exact and obtained approximated solution rises up accordingly. It is clear to see that the table exhibits a slight increase on the error when increasing the N which reduces the complexity. As an overall trend, the approximated solution performs well with small N values.

Example 6. The final example is presented as follow [25]

$$D^a u(x) - D^b u(x) = -1 - e^{x-1}, \qquad 0 < x < 1, \qquad 1 < a \le 2, \qquad 0 < b \le 1, \tag{5.9}$$

$$u(0) = u(1) = 0 \tag{5.10}$$

The exact solution of this problem corresponding to a=2 and b=1 is given by

$$u(x) = x(1 - e^{x-1})$$

In order to solve this problem, we change the value of a (1.5, 1.7 and 2 respectively) while keeping a constant value of b, as implemented in [25]. The main goal of this process is to obtain the approximated solution very close to the exact solution with the value of a approaching 2. Figure-1 presents the comparative results of exact and approximated solutions. The results clearly shows that the approximated solution with a value of 2 achieves a very close solution to the exact one with an average absolute error of $2 * 10^{-3}$ which is a significant performance improvement when compared with the results presented in [25].

6 Discussion

In order to assess the performance of the proposed method, a number of representative examples were solved. We selected a well-known equation, Bagley-Torvik, as the first example which has a polynomial type of exact solution. The following three examples have also different exact solutions

N	β	X=0.1	X=0.3	X=0.5	X=0.7	X=0.9	X=1
2	0.05	0.00004448	0.00039994	0.00110988	0.00217329	0.00358914	0.00442890
	0.02	0.00001200	0.00010796	0.00029983	0.00058754	0.00097102	0.00119866
	0.01	0.00000420	0.00003780	0.00010498	0.00020574	0.00034008	0.00041983
	0.1	0.00012183	0.00109347	0.00302890	0.00591982	0.00975772	0.01202908
	0.2	0.00032666	0.00291545	0.00802908	0.01559820	0.02555064	0.03139724
	0.5	0.00122890	0.01066576	0.02847458	0.05343245	0.08418781	0.10127873
3	0.05	0.00004986	0.00041860	0.00111570	0.00209200	0.00330200	0.00397900
	0.02	0.00001260	0.00011000	0.00029400	0.00055300	0.00087500	0.00105600
	0.01	0.00000451	0.00003910	0.00010435	0.00019619	0.00031051	0.00037483
	0.1	0.00013233	0.00114297	0.00304140	0.00569932	0.00898822	0.01082908
	0.2	0.00035076	0.00302615	0.00804158	0.01504450	0.02367954	0.02849724
	0.5	0.00123085	0.01068227	0.02851771	0.05351159	0.08430972	0.10142373
4	0.05	0.00008215	0.00072184	0.00191435	0.00349191	0.00520607	0.00601469
	0.02	0.00002143	0.00015352	0.00020906	0.00022799	0.00178102	0.00322754
	0.01	0.00000769	0.00006771	0.00018004	0.00032955	0.00049381	0.00057243
	0.1	0.00022503	0.00197690	0.00524303	0.00956596	0.01426917	0.01649208
	0.2	0.00060346	0.00530149	0.01405471	0.02562042	0.03815788	0.04405224
	0.5	0.00209849	0.01848595	0.04911833	0.08969784	0.13375900	0.15447873

TABLE 1. Absolute Error for Example 5.

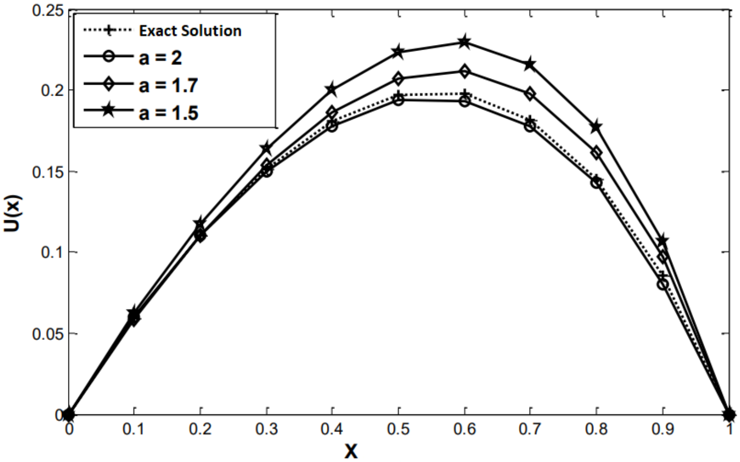

FIGURE 1. Comparison of exact and approximated results for N=4.

in a polynomial form. The proposed method achieved the exact solutions of these examples with respect to the order of the equations. The exact solutions of the first three examples hold a variable with only one degree representing the degree of polynomial. The fourth example has the variable with two degrees. In order to prove the accuracy of the proposed method, the final example has an exponential exact solution in which the proposed method achieved an approximated solution with significantly lower error. It is highly-believed that our method can be applied to solve various equations with the consideration of initial conditions.

7 Conclusion

This work presented a new direction to the field of the solutions of the FDEs which can be easily implemented in practical problems ranging from engineering to economics and natural sciences. The Hermite polynomials are utilized to ensure a non-complex, rapid and accurate solution through the derivation of the operational matrix. Hence, this paper presented a general formulation to derive a new operational matrix of fractional derivatives with Hermite polynomials. The matrix was then integrated with tau method, in order to solve and approximate the linear-type class of FDEs in Caputo sense. A number of illustrative examples has been solved to indicate the advantage of the proposed idea. The results reveal a good accuracy with the exact and approximated solutions. Future work will focus on the adaptation of the proposed strategy to solve the non-linear equations. Also, we will try to solve the systems of FDEs, instead of solving only one FDE. An significant effort will be placed on the solution of some particular FDE, such as Riccati differential equation.

References

[1] D. Baleanu, A. R. Seadawy and A. Ali, *Propagation of harmonic waves in a cylindrical rod via generalized Pochhammer-Chree dynamical wave equation*, Results in Physics **17** (2020) 103039.

[2] D. Lu, A. R. Seadawy and A. Ali, *Applications of exact traveling wave solutions of Modified Liouville and the Symmetric Regularized Long Wave equations via two new techniques*, Results in Physics **9** (2018) 1403-1410.

[3] D. Lu, A. R. Seadawy and A. Ali, *Dispersive traveling wave solutions of the Equal-Width and Modified Equal-Width equations via mathematical methods and its applications*, Results in Physics **9** (2018) 313-320.

[4] D. Lu, A. R. Seadawy and A. Ali, *Structure of traveling wave solutions for some nonlinear models via modified mathematical method*, Open Physics **16** (2018) 854-860.

[5] R. T. Baillie and M. L. King, *Fractional differencing and long memory processes*, Journal of Econometrics **73** (1996).

[6] W. A. Albarakati, A. R. Seadawy and A. Ali, *Analytical wave solutions of the (2+1)-dimensional first integro-differential Kadomtsev-Petviashivili hierarchy equation by using modified mathematical methods*, Results in Physics **15** (2019) 102775.

[7] J. Bisquert, *Fractional diffusion in the multiple-trapping regime and revision of the equivalence with the continuous time random walk*, Physical Review Letters **91** (2003) 010602.

[8] A. G. Butkovskii, S. S. Postnov and E. A Postnova, *Fractional Integro-Differential Calculus and Its Control-Theoretical Applications*, Automation and Remote Control **74** (2013) 543-574.

[9] S. S. Ray and R. K. Bera, *Solution of an extraordinary differential equation by adomian decomposition method*, Journal of Applied Mathematics (2004) 331-338.

[10] M. Dehghan, J. Manafian and A. Saadatmandi, *Solving nonlinear fractional partial differential equations using the homotopy analysis method*, Numerical Methods for Partial Differential Equations **26** (2010) 448-479.

[11] I. Hashim, O. Abdulaziz and S. Momani, *Homotopy analysis method for fractional IVPs*, Communications in Nonlinear Science and Numerical Simulation **14** (2009) 674-684.

[12] S. Odibat, O. Momani and H. Xu, *A reliable algorithm of homotopy analysis method for solving nonlinear fractional differential equations*, Applied Mathematical Modelling **34** (2010) 593-600.

[13] S. Yang, A. Xiao and H. Su, *Convergence of the variational iteration method for solving multi-order fractional differential equations*, Computers & Mathematics with Applications **60** (2010) 2871-2879.

[14] E. H. Doha, A. H. Bhrawy and S. S Ezz-Eldien, *Efficient chebyshev spectral methods for solving multi-term fractional orders differential equations*, Applied Mathematical Modelling **35** (2011) 5662-5672.

[15] A. Saadatmandi and M. Denghan, *A new operational matrix for solving fractional-order differential equations*, Computers & Mathematics with Applications **59** (2010) 1326-1336.

[16] H. S. Roshan, H. Jafari and B. Dumitru, *Solving FDEs with caputo-fabrizio derivative by operational matrix based on genocchi polynomials*, Mathematical Methods in the Applied Sciences **41** (2018) 9134-9141.

[17] E. H. Doha, A. H. Bhrawy and S. S Ezz-Eldien, *A new jacobi operational matrix: an application for solving fractional differential equations*, Applied Mathematical Modelling **36** (2012) 4931-4943.

[18] A. Ahmadian, M. Suleiman and S. Salahshour, *An operational matrix based on legendre polynomials for solving fuzzy fractional-order differential equations*, Abstract and Applied Analysis (2013) 1-29.

[19] A. Bolandpalat, E. Babolian and H. Jafari, *Numerical solutions of multi order fractional differential equations by boubaker polynomials*, Open Physics **14** (2016) 226-230.

[20] G. Dattoli, H. M. Srivastava and K. Zhukovsky, *Orthogonal properties of the Hermite polynomials and related polynomials*, Journal of Computational and Applied Mathematics **182** (2005) 165-172.

[21] Q. M. Mdallal, M. I. Siyam and M. N. Anwar, *A collocation shooting method for solving fractional boundary value problems*, Communications in Nonlinear Sciences and Numerical Solutions **15** (2010) 3814-3822.

[22] M. A. Abdelkawy and T.M. Taha, *An operational matrix of fractional derivatives of laugerre polynomials*, Walailak Journal of Science and Technology **11** (2013) 1041-1055.

[23] S. Kazem, S. Abbasbandy and S. Kumar, *Fractional order Legendre functions for solving fractional-order differential equations*, Applied Mathematical Modelling **37** (2013) 5498-5510.

[24] A. H. Bhrawy, M. Alghamdi and T. M. Taha, *A new modified generalized operational matrix of fractional integration for solving fractional differential equations on the half line*, Advances in Difference Equations (2012).

[25] R. Belgacem, A. Bokhari and A. Amir, *Bernoulli operational matrix of fractional derivative for solution of fractional differential equations*, General Letters in Mathematics **5** (2018) 32-46.