On the Leech dimension of a free partially commutative monoid

Ahmet A. Husainov

Komsomolsk-on-Amur State Technical University, prospect Lenina 27, 681013 Komsomolsk-na-Amure, Russia

E-mail: husainov@knastu.ru

Abstract

We prove that the Leech dimension of any free partially commutative monoid is equal to the supremum of numbers of its mutually commuting generators. As a consequence, we confirm a conjecture that if a free partially commutative monoid does not contain more than n mutually commuting generators, then it is of homological dimension $\leqslant n$. We apply this result to the homological dimension of asynchronous transition systems. We positively answer the question whether the homological dimension of an asynchronous transition system is not greater than the maximal number of its mutually independent events.

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Introduction

Let E be a set and $I \subseteq E \times E$ an irreflexive and symmetric relation. A monoid with a set of generators E and relations $ab = ba$ for all $(a, b) \in I$ is denoted by $M(E, I)$ and called a *free partially commutative monoid*. Generators $a, b \in E$ are called *independent* if $(a, b) \in I$. The present work is devoted to Leech and homological dimension of a free partially commutative monoid.

Our definition is slightly different than the standard definition (cf. [2]). We do not demand that the set E is finite. If E is finite, then $M(E, I)$ is called the *finitely generated free partially commutative monoid*. It was shown in [5] that if the monoid $M(E, I)$ does not contain triples of mutually independent generators, then it is of homological dimension $\leqslant 2$. We conjectured in [6] that if $M(E, I)$ does not contain $(n + 1)$-tuples of mutually independent generators, then it is of homological dimension $\leqslant n$. This conjecture was confirmed in the case of finite E by L. Polyakova [16]. In the present work, we prove the general case. Moreover, we use this result to obtain an estimate for the homological dimensions of asynchronous transition systems. In the paper [5], we conjectured that if an asynchronous transition system does not contain $(n + 1)$-tuples of independent events, then it

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is of homological dimension $\leqslant n$ [5, Open Problem 2]. The conjecture was proved for $n = 1$ and $n = 2$ in [5], and the general case is proved in this paper.

1 Homology groups of small categories

Throughout this paper let Ab be the category of abelian groups and homomorphisms, $\mathbb{Z}$ the additive group of integers, and $\mathbb{N}$ the set of non-negative integers or the free monoid with only one generator. Let pt be the discrete category with one object.

For any category $\mathcal{A}$ and a pair $A_1,\ A_2 \in \operatorname{Ob}\mathcal{A}$, denote by $\mathcal{A}(A_1, A_2)$ the set of all morphisms $A_1 \to A_2$. A *diagram* $\mathscr{C} \to \mathcal{A}$ is a functor from a small category $\mathscr{C}$ to a category $\mathcal{A}$. Given a small category $\mathscr{C}$ we denote by $\mathcal{A}^{\mathscr{C}}$ the category of diagrams $\mathscr{C} \to \mathcal{A}$ and natural transformations.

Let $\Delta_{\mathscr{C}}\,\mathbb{Z}$ be the constant diagram $\mathscr{C} \to \mathrm{Ab}$ with the value $\mathbb{Z}$ at each object and $\Delta_{\mathscr{C}}\,\mathbb{Z}(\alpha) = 1_{\mathbb{Z}}$ for all $\alpha \in \operatorname{Mor}(\mathscr{C})$. If the category $\mathscr{C}$ is clear from the context, we write $\Delta\,\mathbb{Z}$.

Since Leech dimension of a monoid is defined using homology of the opposite category of the factorization category, we recall some results from the homology theory of categories, and consider some properties of the factorization category. We consider homology groups of small categories $\mathscr{C}$ with coefficients in diagrams $F : \mathscr{C} \to \mathrm{Ab}$ and describe their connection with the left derived of the colimit functor. Using Oberst's theorem about isomorphisms $\varinjlim_n^{\mathscr{C}^{\mathrm{op}}}(F \circ S) \to \varinjlim_n^{\mathscr{D}^{\mathrm{op}}} F$ in the case of a strongly coinitial functor $S : \mathscr{C} \to \mathscr{D}$, we give an approach to estimating the homological dimension of a small category.

1.1 Homology of categories and derived functors of colimit

Let $\mathscr{C}$ be a small category and $F : \mathscr{C} \to \mathrm{Ab}$ a diagram. We consider the chain complex $C_*(\mathscr{C}, F)$ whose n-term is zero for $n < 0$, and for $n \geqslant 0$ is given by

$$C_n(\mathscr{C}, F) = \bigoplus_{c_0 \to \cdots \to c_n} F(c_0)$$

where the coproduct ranges over all n-fold sequences of compatible morphisms in $\mathscr{C}$. For each $c_0 \overset{\alpha_1}{\to} c_1 \overset{\alpha_2}{\to} \cdots \overset{\alpha_n}{\to} c_n$, we consider the corresponding direct summand of $C_n(\mathscr{C}, F)$ as the abelian group of the pairs $(c_0 \to \cdots \to c_n, x)$ with $x \in F(c_0)$. Define the *boundary operators* by

$$d_n = \sum_{i=0}^{n} (-1)^i d_i^n : C_n(\mathscr{C}, F) \to C_{n-1}(\mathscr{C}, F)$$

where d_i^n is the homomorphism which assigns to each

$$(c_0 \overset{\alpha_1}{\to} c_1 \overset{\alpha_2}{\to} \cdots \overset{\alpha_n}{\to} c_n, x) \in \bigoplus_{c_0 \to \cdots \to c_n} F(c_0)$$

the element

$$\begin{cases} (c_1 \stackrel{\alpha_2}{\to} \cdots \stackrel{\alpha_n}{\to} c_n, F(c_0 \stackrel{\alpha_1}{\to} c_1)(x)), & \text{if } i = 0 \\ (c_0 \stackrel{\alpha_1}{\to} \cdots \stackrel{\alpha_{i-1}}{\to} c_{i-1} \stackrel{\alpha_{i+1}\alpha_i}{\to} c_{i+1} \stackrel{\alpha_{i+2}}{\to} \cdots \stackrel{\alpha_n}{\to} c_n, x), & \text{if } 1 \leqslant i \leqslant n-1 \\ (c_0 \stackrel{\alpha_1}{\to} \cdots \stackrel{\alpha_{n-1}}{\to} c_{n-1}, x), & \text{if } i = n \end{cases}$$

Definition 1.1. For $n \geqslant 0$, the group $H_n(C_*(\mathscr{C}, F)) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}$ is denoted by $H_n(\mathscr{C}, F)$ and called the *n-th homology group of $\mathscr{C}$ with coefficients in a diagram F.*

It is well known [3, Appl. 2] that the functors $H_n(\mathscr{C}, -) : \mathrm{Ab}^{\mathscr{C}} \to \mathrm{Ab}$ make up the connected sequence of the left satellites of the colimit functor $\varinjlim^{\mathscr{C}} : \mathrm{Ab}^{\mathscr{C}} \to \mathrm{Ab}$. Since $\mathrm{Ab}^{\mathscr{C}}$ is the abelian category with enough projectives, these satellites are isomorphic to the left derived functors of $\varinjlim^{\mathscr{C}} : \mathrm{Ab}^{\mathscr{C}} \to \mathrm{Ab}$. For any $n \geqslant 0$, denote by $\varinjlim_n^{\mathscr{C}} : \mathrm{Ab}^{\mathscr{C}} \to \mathrm{Ab}$ the n-th left satellite of the colimit.

The group $H_n(\mathscr{C}) =_{def} H_n(\mathscr{C}, \Delta_{\mathscr{C}} \mathbb{Z})$ will be called the *n-th integer homology group* of $\mathscr{C}$. It follows from the Eilenberg Theorem [3, Appl. 2] that $H_n(\mathscr{C})$ is isomorphic to the homology groups of geometric realization of the nerve of $\mathscr{C}$. For example,

$$H_n(\mathrm{pt}) = \begin{cases} \mathbb{Z}, & \text{if} \quad n = 0, \\ 0, & \text{if} \quad n > 0. \end{cases}$$

Definition 1.2. A small category $\mathscr{C}$ is said to be *acyclic* if for all integer $n \geqslant 0$ there exist isomorphisms $H_n(\mathscr{C}) \cong H_n(\mathrm{pt})$.

1.2 Strong coinitiality

Let $S : \mathscr{C} \to \mathscr{D}$ be a functor from a small category to a category. Let $d \in \operatorname{Ob} \mathscr{D}$. The *comma category* (cf. [13, § II.6]) S/d is the category with objects the pairs (c, α) with $c \in \operatorname{Ob} \mathscr{C}$ and $\alpha \in \mathscr{D}(S(c), d)$ and with morphisms $(c_1, \alpha_1) \to (c_2, \alpha_2)$ the triples (f, α_1, α_2) for which $f \in \mathscr{C}(c_1, c_2)$ and $\alpha_2 \circ S(f) = \alpha_1$. If S is full imbedding $\mathscr{C} \subseteq \mathscr{D}$, then S/d is denoted by $\mathscr{C}/d$. A functor $S : \mathscr{C} \to \mathscr{D}$ between small categories is said to be *strongly coinitial* if for every object $d \in \mathscr{D}$ the integer homology group $H_n(S/d)$ are isomorphic to $H_n(\mathrm{pt})$. It occurs if and only if for each $d \in \mathscr{D}$

$$H_n(S/d) = \begin{cases} 0\,, & \text{if} \quad n > 0 \\ \mathbb{Z}\,, & \text{if} \quad n = 0\,. \end{cases}$$

The *comma category* d/S is the category with objects the pairs (c, α) with $c \in \operatorname{Ob}(\mathscr{C})$ and $\alpha \in \mathscr{D}(d, S(c))$. Its morphisms $(c_1, \alpha_1) \to (c_2, \alpha_2)$ are the triples (f, α_1, α_2) with $f \in \mathscr{C}(c_1, c_2)$ satisfying $S(f) \circ \alpha_1 = \alpha_2$. Since $\mathscr{C}$ and $\mathscr{C}^{\mathrm{op}}$ have homeomorphic geometric realizations of the nerves, we can prove the following lemma.

Lemma 1.3. Let $\mathscr{C}$ be a small category. There are isomorphisms $H_n(\mathscr{C}^{\mathrm{op}}) \cong H_n(\mathscr{C})$ for all $n \geqslant 0$.

Proposition 1.4. Let $S : \mathscr{C} \to \mathscr{D}$ be a functor between small categories. For every $n \geqslant 0$, there are canonical homomorphisms $\varinjlim_n^{\mathscr{C}^{\mathrm{op}}} (F \circ S^{\mathrm{op}}) \to \varinjlim_n^{\mathscr{D}^{\mathrm{op}}} F$ which are natural in F. These homomorphisms are isomorphisms for all $n \geqslant 0$ and $F : \mathscr{D}^{\mathrm{op}} \to \mathrm{Ab}$ if and only if the functor S is strongly coinitial.

Proof. In [15], Oberst constructed homomorphisms $\eta_F^n : \varinjlim_n^{\mathscr{C}} (F \circ S) \to \varinjlim_n^{\mathscr{D}} F$, natural in $F \in \mathrm{Ab}^{\mathscr{D}}$. These η_F^n are isomorphisms for all $n \geqslant 0$ and $F \in \mathrm{Ab}^{\mathscr{D}}$ if and only if the categories d/S are acyclic for all $d \in \mathrm{Ob}\,\mathscr{D}$. If we substitute S by $S^{\mathrm{op}} : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}^{\mathrm{op}}$ and apply [15, Theorem 2.3], then we obtain that the homomorphisms η_F^n are isomorphisms if and only if the categories $(S/d)^{\mathrm{op}}$ are acyclic. Homology groups $\varinjlim_n^{(S/d)^{\mathrm{op}}} \Delta\,\mathbb{Z}$ and $\varinjlim_n^{S/d} \Delta\,\mathbb{Z}$ are isomorphic by Lemma 1.3. Consequently, this condition is equivalent to acyclic property of S/d. Hence the homomorphisms η_F^n are isomorphisms if and only if S is strongly coinitial. Q.E.D.

1.3 The category of factorizations

Let $\mathscr{C}$ be a small category. We denote by $\mathfrak{F}\mathscr{C}$ the *category of factorization in* $\mathscr{C}$ (cf. [1]). Recall that $\mathrm{Ob}(\mathfrak{F}\mathscr{C}) = \mathrm{Mor}(\mathscr{C})$ and for every $\alpha, \beta \in \mathrm{Ob}(\mathfrak{F}\mathscr{C})$ the set of morphisms $\mathfrak{F}\mathscr{C}(\alpha, \beta)$ consists of the pairs (f, g) of morphisms $f, g \in \mathrm{Mor}\,\mathscr{C}$

$$\begin{array}{ccc} \circ & \xrightarrow{\quad g \quad} & \circ \\ \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\beta} \\ \circ & \xleftarrow{\quad f \quad} & \circ \end{array}$$

satisfying $g \circ \alpha \circ f = \beta$. The composition of $\alpha \overset{(f_1,g_1)}{\to} \beta$ and $\beta \overset{(f_2,g_2)}{\to} \gamma$ is defined by $\alpha \overset{(f_1 \circ f_2, g_2 \circ g_1)}{\longrightarrow} \gamma$. The identity morphism of an object $a \overset{\alpha}{\to} b$ in $\mathfrak{F}\mathscr{C}$ is the pair of the identity morphisms $\alpha \overset{(1_a,1_b)}{\longrightarrow} \alpha$.

Lemma 1.5. For any category $\mathscr{C}$ there is an isomorphism $\mathfrak{F}(\mathscr{C}^{\mathrm{op}}) \cong \mathfrak{F}\mathscr{C}$.

Corollary 1.6. Let $\mathscr{C}$ be a category and α one of its morphisms. Then $\mathfrak{F}\mathscr{C}/\alpha$ is isomorphic to the category defined as follows: The objects are triples of morphisms (x, β, y) for which the composition is defined and $x \circ$

$\beta \circ y = \alpha$. The morphisms $(x, \beta, y) \to (x', \beta', y')$ are commutative diagrams

$$\begin{array}{ccccc} & \circ & \xleftarrow{\beta} & \circ & \\ {}^{x}\swarrow & \downarrow v & & \uparrow w & \nwarrow{}^{y} \\ \circ & & & & \circ \\ {}_{x'}\nwarrow & \downarrow & & \uparrow & \swarrow{}_{y'} \\ & \circ & \xleftarrow{\beta'} & \circ & \end{array}$$

In [1], Baues and Wirsching introduced the cohomology of categories with coefficients in a natural system. It follows from [1, Theorem 4.4] that this cohomology can be defined as the right derived functor $\varprojlim^n_{\mathfrak{F}\mathscr{C}}$ of limit. We give a dual definition.

Definition 1.7. Let $\mathscr{C}$ be a small category and $\mathfrak{F}\mathscr{C}$ the category of factorizations. A *contravariant natural system on* $\mathscr{C}$ is a functor $F : (\mathfrak{F}\mathscr{C})^{\mathrm{op}} \to$ Ab. For $n \geqslant 0$, the *n-th homology group of* $\mathscr{C}$ *with coefficients in* F is the abelian groups $\varinjlim_n^{(\mathfrak{F}\mathscr{C})^{\mathrm{op}}} F$.

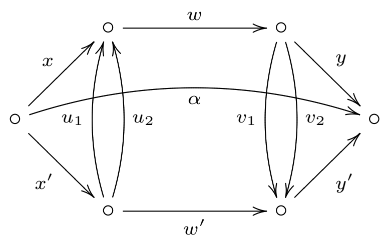
Since the category of factorization has a complicated structure, we seek its strong coinitial subcategories with a simple construction. For that we shall study comma categories $\mathfrak{F}\mathscr{C}/\alpha$.

A category is said to be *cancellative* if its morphisms are epimorphic and monomorphic. A category $\mathscr{C}$ is cancellative if and only if the implications $\alpha\gamma = \beta\gamma \;\Rightarrow\; \alpha = \beta$ and $\gamma\alpha = \gamma\beta \;\Rightarrow\; \alpha = \beta$ are true for all $\alpha, \beta, \gamma \in \mathscr{C}$. Lemma 1.8 is useful for cancellative categories.

Recall that a preordered set may be defined as a small category $\mathscr{C}$ such that for any $a, b \in \mathrm{Ob}\,\mathscr{C}$ the set $\mathscr{C}(a, b)$ is either empty or has precisely one element. We write $a \leqslant b$ in the later case. If moreover $a \leqslant b$ and $b \leqslant a$ implies $a = b$, then $\mathscr{C}$ is the partially ordered set. A morphism α is called a *retraction* if there are $\beta \in \mathrm{Mor}\,\mathscr{C}$ and $c \in \mathrm{Ob}\,\mathscr{C}$ satisfying condition $\alpha \circ \beta = 1_c$.

Lemma 1.8. Let $\mathscr{C}$ be a small cancellative category. If $\mathscr{C}$ does not contain nonidentity retractions, then $\mathfrak{F}\mathscr{C}/\alpha$ is a partially ordered set for each $\alpha \in$ *Ob*$(\mathfrak{F}\mathscr{C})$.

Proof. For any $\alpha \in \mathrm{Mor}\,\mathscr{C}$, the objects of the category $\mathfrak{F}\mathscr{C}/\alpha$ are triples of morphisms (x, w, y) of $\mathscr{C}$ for which $y \circ w \circ x = \alpha$. Consider any (parallel) pair of morphisms $(x, w, y) \underset{(u_2,v_2)}{\overset{(u_1,v_1)}{\rightrightarrows}} (x', w', y')$ of the category $\mathfrak{F}\mathscr{C}/\alpha$. These morphisms are given by the commutative diagram



It follows from $y'v_1 = y = y'v_2$ that $v_1 = v_2$. Similarly, $u_1x' = x = u_2x'$ implies $u_1 = u_2$. Hence $u_1 = u_2$ and $v_1 = v_2$. Consequently, any two morphisms $(x, w, y) \to (x', w', y')$ are same. It follows that $\mathfrak{F}\mathscr{C}/\alpha$ is the preordered set. If $\mathscr{C}$ does not contain nonidentity retraction, then any two morphisms

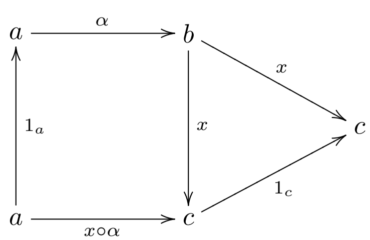
$$(x, w, y) \underset{(u_2,v_2)}{\overset{(u_1,v_1)}{\rightleftarrows}} (x', w', y')$$

lead to the equalities $u_1x' = x$, $u_2x = x'$, $y'v_1 = y$, $yv_2 = y'$. It follows that $u_1u_2x = x$, $yv_2v_1 = y$. Since $\mathscr{C}$ is cancellative, we obtain that u_1u_2 and v_2v_1 are identity morphisms. If $\mathscr{C}$ does not contain nonidentity retraction, then it results from this that u_1, u_2, v_1, v_2 are identity morphisms. Therefore the relation $\leqslant$ is antisymmetric and $\mathfrak{F}\mathscr{C}/\alpha$ is the partially ordered set. Q.E.D.

Let $\mathscr{C}$ be a small category. Consider the functor $\mathfrak{F}\mathscr{C} \xrightarrow{t} \mathscr{C}$ which assigns to every morphism $a \to b$ its codomain b and to every morphism $\alpha \overset{(f,g)}{\to} \beta$ the morphism $g : t(\alpha) \to t(\beta)$. The following assertion was implicitly formulated and proved in [9].

Lemma 1.9. The functor $\mathfrak{F}\mathscr{C} \xrightarrow{t} \mathscr{C}$ is strongly coinitial.

Proof. For each $c \in \mathscr{C}$, the category t/c contains the full subcategory with objects $(\beta, 1_c)$. This subcategory is isomorphic to $(\mathscr{C}/c)^{\text{op}}$. For each object $(\alpha, x) \in Ob(t/c)$ there is the morphism $(\alpha, x) \to (x \circ \alpha, 1_c)$ which is defined by the commutativity of the diagram



This morphism has the following universal property: For any morphism $(\alpha, x) \to (\beta, 1_c)$ there is an unique morphism $(x \circ \alpha, 1_c) \to (\beta, 1_c)$ making the diagram

$$\begin{array}{ccc} (\alpha, x) & \longrightarrow & (x \circ \alpha, 1_c) \\ & \searrow \quad \swarrow_{\exists !} & \\ & (\beta, 1_c) & \end{array}$$

to be commutative. It follows that the full inclusion $(\mathscr{C}/c)^{\mathrm{op}} \subseteq t/c$ has a left adjoint functor $\sigma : t/c \to (\mathscr{C}/c)^{\mathrm{op}}$. Hence the inclusion $\mathscr{C}/c \subseteq (t/c)^{\mathrm{op}}$ is strongly coinitial and $\varinjlim_n^{t/c} \Delta\, \mathbb{Z} \circ \sigma \cong \varinjlim_n^{(\mathscr{C}/c)^{\mathrm{op}}} \Delta\, \mathbb{Z}$. Therefore $H_q(t/c) = 0$ for $q > 0$, and $H_0(t/c) \cong \mathbb{Z}$. Q.E.D.

Proposition 1.4 yields the following corollary:

Corollary 1.10. There are isomorphisms $\varinjlim_n^{(\mathfrak{F}\mathscr{C})^{\mathrm{op}}} (F \circ t^{\mathrm{op}}) \cong \varinjlim_n^{\mathscr{C}^{\mathrm{op}}} F$ for all diagrams $F : \mathscr{C}^{\mathrm{op}} \to \mathrm{Ab}$ and $n \geqslant 0$.

This implies in particular that $H_n((\mathfrak{F}\mathscr{C})^{\mathrm{op}}) \cong H_n(\mathscr{C})$ for all $n \geqslant 0$.

1.4 Homological dimension of small categories

Let $\mathbb{N}$ be the set of all nonnegative integers. Consider $\mathbb{N}$ as the subset of $\{-1\} \cup \mathbb{N} \cup \{\infty\}$ ordered by $-1 < 0 < 1 < 2 < \cdots < \infty$.

Definition 1.11. Let $\mathscr{C}$ be a small category. The *homological dimension* $\mathrm{hd}\,\mathscr{C}$ is the sup of $n \in \mathbb{N}$ for which $\varinjlim_n^{\mathscr{C}} \neq 0$.

For example, $\mathrm{hd}\,\mathscr{C} = -1$ if and only if $\mathscr{C} = \varnothing$. The category pt has the homological dimension $\mathrm{hd}\,\mathrm{pt} = 0$. For a comprehensive survey of the homological dimension of small categories, we refer the reader to [4].

Proposition 1.4 implies the following assertion for estimation of the homological dimension:

Corollary 1.12. If there is a strongly coinitial functor $S : \mathscr{C} \to \mathscr{D}$ between small categories, then $\mathrm{hd}\,\mathscr{C}^{\mathrm{op}} \geqslant \mathrm{hd}\,\mathscr{D}^{\mathrm{op}}$.

A subcategory $\mathscr{D} \subseteq \mathscr{C}$ is said to be *open* if $\mathscr{D}$ is a full subcategory containing the codomain for any morphism whose domain is in $\mathscr{D}$. A subcategory $\mathscr{D} \subseteq \mathscr{C}$ is *closed* if $\mathscr{D}^{\mathrm{op}}$ is open in $\mathscr{C}^{\mathrm{op}}$.

Let $\mathscr{D} \subseteq \mathscr{C}$ be a closed subcategory. The *restriction functor* $\mathrm{Ab}^{\mathscr{C}^{\mathrm{op}}} \to \mathrm{Ab}^{\mathscr{D}^{\mathrm{op}}}$ is given by $G \mapsto G|_{\mathscr{D}^{\mathrm{op}}}$. It has the left adjoint *functor of adding zeros* $\mathrm{Ab}^{\mathscr{D}^{\mathrm{op}}} \to \mathrm{Ab}^{\mathscr{C}^{\mathrm{op}}}$ which assigns to each diagram $F \in \mathrm{Ab}^{\mathscr{D}^{\mathrm{op}}}$ the diagram $F_{\mathscr{D}}$ defined by $F_{\mathscr{D}}|_{\mathscr{D}^{\mathrm{op}}} = F$ and $F_{\mathscr{D}}(c) = 0$ for $c \notin \mathrm{Ob}(\mathscr{D})$. The diagram $F_{\mathscr{D}}$ is said to be *obtained by adding zeros*. It is easy to see that both the restriction functor and the functor of adding zeros are exact.

Lemma 1.13. Let $\mathscr{D} \subseteq \mathscr{C}$ be a closed subcategory. There are isomorphisms $\varinjlim_n^{\mathscr{C}^{\mathrm{op}}} F_{\mathscr{D}} \cong \varinjlim_n^{\mathscr{D}^{\mathrm{op}}} F$ for all $F \in \mathrm{Ab}^{\mathscr{D}^{\mathrm{op}}}$ and $n \geqslant 0$.

Proof. Consider an arbitrary projective resolution of the diagram $F \in \mathrm{Ab}^{\mathscr{D}^{\mathrm{op}}}$

$$0 \leftarrow F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow \cdots .$$

Since $\varinjlim_n^{\mathscr{D}^{\mathrm{op}}}$ are the derived functors of the colimit, the homology groups of the chain complex $\varinjlim^{\mathscr{D}^{\mathrm{op}}} F_*$ are isomorphic to $\varinjlim_n^{\mathscr{D}^{\mathrm{op}}} F$. On the other hand, the functor of adding zeros, as a left adjoint to an exact functor, assigns to the resolution F_* the projective resolution $F_{*\mathscr{D}}$ of $F_{\mathscr{D}}$. Since $\varinjlim^{\mathscr{C}^{\mathrm{op}}} F_{\mathscr{D}} \cong \varinjlim^{\mathscr{D}^{\mathrm{op}}} F$, the complex $\varinjlim^{\mathscr{C}^{\mathrm{op}}} F_{*\mathscr{D}}$ is isomorphic to the complex $\varinjlim^{\mathscr{D}^{\mathrm{op}}} F_*$. It follows that the homology groups of these complexes are isomorphic. Hence $\varinjlim_n^{\mathscr{C}^{\mathrm{op}}} F_{\mathscr{D}} \cong \varinjlim_n^{\mathscr{D}^{\mathrm{op}}} F$. Q.E.D.

Lemma 1.14. Let $\mathscr{C}$ be a small category. If $\mathscr{D} \subseteq \mathscr{C}$ is an open subcategory, then $\mathrm{hd}\,\mathscr{D} \leqslant \mathrm{hd}\,\mathscr{C}$. If $\bigcup_{j \in J} \mathscr{C}_j$ is the union of open subcategories of $\mathscr{C}$, then

$$\mathrm{hd} \bigcup_{j \in J} \mathscr{C}_j = \sup_{j \in J} \{\mathrm{hd}\,\mathscr{C}_j\}.$$

Proof. For any $a \in \mathrm{Ob}\,\mathscr{C}$, denote by $\mathscr{C}^a \subseteq \mathscr{C}$ the smallest open subcategory of $\mathscr{C}$ containing a. By Mitchell [14, Corollary 10], $\mathrm{hd}\,\mathscr{C} = \sup_{a \in \mathrm{Ob}\mathscr{C}} \{\mathrm{hd}\,\mathscr{C}^a\}$. If $\mathscr{D}$ is an open subcategory, then $\mathscr{D} = \bigcup_{d \in \mathrm{Ob}(\mathscr{D})} \mathscr{C}^d$. It follows that $\mathrm{hd}\,\mathscr{D} \leqslant \mathrm{hd}\,\mathscr{C}$. If all subcategories $\mathscr{C}_j$ are open, then $\mathscr{C}^a = \mathscr{C}_j^a$ for every $a \in \mathscr{C}_j$, and the second assertion follows from

$$\sup_j \left\{ \sup_{a \in \mathrm{Ob}\mathscr{C}_j} \{\mathrm{hd}\,\mathscr{C}^a\} \right\} = \sup_{a \in \bigcup_{j \in J} \mathrm{Ob}\mathscr{C}_j} \{\mathrm{hd}\,\mathscr{C}^a\}.$$

Q.E.D.

Lemma 1.15. Let $\mathscr{C}$ and $\mathscr{D}$ be small categories. Then $\mathrm{hd}(\mathscr{C} \times \mathscr{D}) \leqslant \mathrm{hd}\,\mathscr{C} + \mathrm{hd}\,\mathscr{D}$.

Proof. For any abelian category $\mathcal{A}$ with exact products and functor $F : \mathscr{C} \times \mathscr{D} \to \mathcal{A}$ there is the first quadrant spectral sequence with $E_2^{p,q} = \varprojlim^p_{c \in \mathscr{C}} \varprojlim^q_{d \in \mathscr{D}} \{F(c,d)\}$ converging to $\varprojlim^{p+q}_{\mathscr{C} \times \mathscr{D}} F$ [8]. Substitute $\mathscr{C}^{\mathrm{op}}$ and $\mathscr{D}^{\mathrm{op}}$ instead of $\mathscr{C}$ and $\mathscr{D}$. Since $\mathcal{A} = \mathrm{Ab}^{\mathrm{op}}$ has exact products, we have the spectral sequence $\varinjlim_p^{c \in \mathscr{C}} \varinjlim_q^{d \in \mathscr{D}} \{F(c,d)\} \Rightarrow \varinjlim_{p+q}^{\mathscr{C} \times \mathscr{D}} F$ giving the relation $\mathrm{hd}(\mathscr{C} \times \mathscr{D}) \leqslant \mathrm{hd}\,\mathscr{C} + \mathrm{hd}\,\mathscr{D}$. Q.E.D.

Since $H_n(\mathscr{C})$ is isomorphic to the homology groups of geometric realization of the nerve, we get the Künneth formula:

Lemma 1.16 (Künneth formula)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be small categories. For any $n \geqslant 0$, the group $H_n(\mathscr{C} \times \mathscr{D})$ is isomorphic to

$$\left(\bigoplus_{p+q=n} H_p(\mathscr{C}) \otimes H_q(\mathscr{D})\right) \oplus \left(\bigoplus_{p+q=n-1} \operatorname{Tor}\left(H_p(\mathscr{C}), H_q(\mathscr{D})\right)\right).$$

1.5 Acyclic partially ordered sets

Recall that a small category $\mathscr{C}$ is *acyclic* if $H_n(\mathscr{C}) \cong H_n(\mathrm{pt})$ for all $n \geqslant 0$. The study of strongly coinitial functors is closely related with the property of acyclicity. There is a simple condition for a partially ordered set that enforces that it is acyclic.

Lemma 1.17. Let X be a partially ordered set. Let V and W be its closed subsets for which $X = V \cup W$. If V, W, and $V \cap W$ are acyclic, then X is acyclic.

Proof. Denote $C_*(X, \Delta_X \mathbb{Z})$ by $C_*(X)$. The unique functor $X \to \mathrm{pt}$ gives the chain homomorphism $C_*(X) \to C_*(\mathrm{pt})$. Denote by $\tilde{C}_*(X)$ its kernel. It is easy to see that X is acyclic if and only if $H_n(\tilde{C}_*(X)) = 0$ for all $q \geqslant 0$. The subsets V^{op} and W^{op} are open and $X^{\mathrm{op}} = V^{\mathrm{op}} \cup W^{\mathrm{op}}$. There exists a exact sequence

$$0 \to \tilde{C}_*(V^{\mathrm{op}} \cap W^{\mathrm{op}}) \to \tilde{C}_*(V^{\mathrm{op}}) \oplus \tilde{C}_*(W^{\mathrm{op}}) \to \tilde{C}_*(X^{\mathrm{op}}) \to 0.$$

A corresponding long exact sequence leads us to $H_n(\tilde{C}_*(X^{\mathrm{op}})) = 0$ for all $q \geqslant 0$. Q.E.D.

An easy induction proves the following corollary.

Corollary 1.18. Let $X = \bigcup_{i=1}^{n} W_i$ be the union of closed subsets $W_i \subseteq X$. If the intersections $W_{i_1} \cap W_{i_2} \cap \cdots \cap W_{i_k}$ are acyclic for all $\{i_1, i_2, \cdots, i_k\} \subseteq \{1, 2, \cdots, n\}$, then X is acyclic.

2 Category of factorizations of a free partially commutative monoid

In this section, we prove that the full subcategory consisting of all finite products of mutually independent generators is strongly coinitial in the category of factorization $\mathfrak{F}M(E, I)$.

Let $M(E, I)$ be a free partially commutative monoid with a set of generators E and relations $ab = ba$ for all $(a, b) \in I$. Denote by $E_v \subseteq E$, $v \in V$, the maximal subsets consisting of its mutually independent generators. We write $M(E_v) \subseteq M(E, I)$ for the monoid generated by E_v.

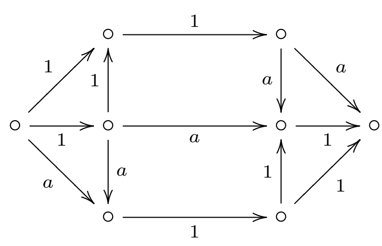
The union $\bigcup_{v\in V} M(E_v) \subseteq M(E,I)$ is not a subcategory. This creates difficulties for studying the homology of $M(E,I)$. The situation can be saved by considering the inclusion of the corresponding categories of factorizations: The full subcategory of $\mathfrak{F}M(E,I)$ with the set of objects $M(E_v)$ is equal to $\mathfrak{F}M(E_v)$. Moreover, the subcategories $\mathfrak{F}M(E_v) \subseteq \mathfrak{F}M(E,I)$ are closed.

First we study the comma category $\mathfrak{F}M(E,I)/\alpha$ for any element $\alpha \in M(E,I)$.

Proposition 2.1. Let $M(E,I)$ be a free partially commutative monoid and $\alpha \in M(E,I)$ an element. Then the category $\mathfrak{F}M(E,I)/\alpha$ is a partially ordered set.

Proof. If E is finite, then the monoid $M(E,I)$ is cancellative by [2, Corollary 2]. Let E be infinite. Suppose that $yw_1x = yw_2x$. There is a finite subset $E' \subseteq E$ consisting of symbols which are multipliers of the elements $x, y, w_1, w_2 \in M(E,I)$. Since E' is finite, the submonoid $M(E', I \cap (E' \times E'))$ is cancellative. It follows that $w_1 = w_2$. Hence $M(E,I)$ is cancellative. We assign to every $x \in M(E,I)$ a length $|x|$ of a word which presents x and call this the *length* of x. It easy to see that $|xy| = |x| + |y|$ for all $x, y \in M(E,I)$. Therefore $M(E,I)$ does not contain nonidentity retractions. By Lemma 1.8 we conclude that $\mathfrak{F}M(E,I)/\alpha$ is a partially ordered set. Q.E.D.

Example 2.2. Let $M(E,I)$ be a free partially commutative monoid and $a \in E$ a generator. Describe the partially ordered set $\mathfrak{F}M(E,I)/a$. It consists of the elements $(1,1,a)$, $(1,a,1)$, and $(a,1,1)$. Its morphisms are defined by the diagram



The morphisms $(1,1,a) \overset{(1,a)}{\to} (1,a,1)$ and $(a,1,1) \overset{(a,1)}{\to} (1,a,1)$ leads us to the relation $(1,1,a) < (1,a,1) > (a,1,1)$ in the partially ordered set $\mathfrak{F}M(E,I)/a$.

Two elements $x, y \in M(E,I)$ are said to be *commuting* if $xy = yx$. We notice attention that generators a and b are commuting if and only if $(a,b) \in I$ or $a = b$.

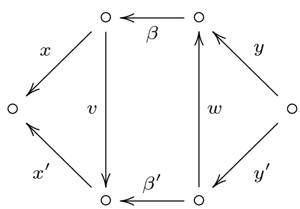
The category $\mathfrak{F}M(E,I)$ contains the full subcategory $\bigcup\limits_{v\in V}\mathfrak{F}M(E_v)$ whose objects are products of the pairwise commuting generators.

Theorem 2.3. The inclusion $\bigcup\limits_{v\in V}\mathfrak{F}M(E_v)\subseteq\mathfrak{F}M(E,I)$ is strongly coinitial.

Proof. By Lemma 1.5, the category $\mathfrak{F}M(E,I)$ is isomorphic to $\mathfrak{F}(M(E,I)^{\mathrm{op}})$. This isomorphism carries the subcategory $\bigcup\limits_{v\in V}\mathfrak{F}M(E_v)$ to $\bigcup\limits_{v\in V}\mathfrak{F}(M(E_v)^{\mathrm{op}})$. Thus, it is enough to prove that the inclusion of $\bigcup\limits_{v\in V}\mathfrak{F}(M(E_v)^{\mathrm{op}})$ into $\mathfrak{F}(M(E,I)^{\mathrm{op}})$ is strongly coinitial. This is equivalent to the assertion that for each $\alpha\in M(E,I)$

$$H_q(\bigcup_{v\in V}\mathfrak{F}(M(E_v)^{\mathrm{op}})/\alpha)\cong H_q(\mathrm{pt}),\text{for all } q\in\mathbb{N}\ .$$

We prove this assertion by induction on the length $|\alpha|$. By Proposition 2.1, the comma category $\mathfrak{F}(M(E,I)^{\mathrm{op}})/\alpha$ is a partially ordered set. By Corollary 1.6, its elements may be given as triples (x,β,y) of elements of $M(E,I)$ satisfying $x\circ\beta\circ y=\alpha$. The relation $(x,\beta,y)\leqslant(x',\beta',y')$ is defined by existence of pairs (v,w) making the following diagram to be commutative



The subcategory $(\bigcup\limits_{v\in V}\mathfrak{F}(M(E_v)^{\mathrm{op}}))/\alpha\cong\bigcup\limits_{v\in V}(\mathfrak{F}(M(E_v)^{\mathrm{op}})/\alpha)$ is its closed subset consisting of the triples (x,β,y) for which $\beta\in\bigcup\limits_{v\in V}M(E_v)$.

Let $\Phi(\alpha)=\bigcup\limits_{v\in V}(\mathfrak{F}(M(E_v)^{\mathrm{op}})/\alpha)$. We shall prove by induction on $|\alpha|$ that the partially ordered set $\Phi(\alpha)$ is acyclic. We use that $\Phi(\alpha)$ is union of closed subsets which are isomorphic to $\Phi(\beta)$ with $|\beta|<|\alpha|$.

If $|\alpha|=0$, then $\alpha=1$, hence the partially ordered set $\Phi(\alpha)$ consists of unique element $(1,1,1)$. So, the assertion is true in this case.

Suppose that for every $\beta\in M(E,I)$ with $|\beta|<n$ there are isomorphisms $H_q(\Phi(\beta))\cong H_q(\mathrm{pt})$ for all $q\geqslant 0$. Prove it for α with $|\alpha|=n$.

If $\alpha=f\circ g$, then we say that f is a *left divisor* of $\alpha\in M(E,I)$ and g is its *right divisor*. In this case, there are injections of the partially ordered sets

$$\mathfrak{F}(M(E,I)^{\mathrm{op}})/g\xrightarrow{f_*}\mathfrak{F}(M(E,I)^{\mathrm{op}})/\alpha\xleftarrow{g^*}\mathfrak{F}(M(E,I)^{\mathrm{op}})/f,$$

where $f_*(x,\beta,y) = (f\circ x,\beta,y)$, $g^*(x,\beta,y) = (x,\beta,y\circ g)$. Images of the maps

$$\Phi(g) \xrightarrow{f_*} \Phi(\alpha) \xleftarrow{g^*} \Phi(f)$$

are closed in $\Phi(\alpha)$. In general, if $f\circ\gamma\circ g = \alpha$ for some $f,g,\gamma \in M(E,I)$, then the image of the inclusion

$$f_*g^* = g^*f_* : \Phi(\gamma) \longrightarrow \Phi(\alpha)$$

is closed in $\Phi(\alpha)$.

Let $\{a_1,\cdots,a_m\} \subseteq E$ be the set of generators which are left divisors of α and $\{b_1,\cdots,b_n\} \subseteq E$ its right divisors. It is easy to see that the generators $a_1,\cdots,a_m$ are mutually independent. Similarly $b_1,\cdots,b_n$ are mutually independent. Consider the following closed subsets of $\Phi(\alpha)$

$$L_i(\alpha) = \{(x,\beta,y) \in \Phi(\alpha) : x \text{ has the left divisor } a_i\},$$

$$R_j(\alpha) = \{(x,\beta,y) \in \Phi(\alpha) : y \text{ has the right divisor } b_j\},$$

where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. Let $L(\alpha) = \bigcup\limits_{i=1}^{m} L_i(\alpha)$ and $R(\alpha) = \bigcup\limits_{j=1}^{n} R_j(\alpha)$. Evidently, we have that $L(\alpha) = \{(x,\beta,y) : \beta \in \bigcup\limits_{v\in V} M(E_v), x \neq 1\}$ and $R(\alpha) = \{(x,\beta,y) : \beta \in \bigcup\limits_{v\in V} M(E_v), y \neq 1\}$.

Consider the following two cases: $\alpha \in \bigcup\limits_{v\in V} M(E_v)$ and $\alpha \notin \bigcup\limits_{v\in V} M(E_v)$. If $\alpha \in \bigcup\limits_{v\in V} M(E_v)$, then $\Phi(\alpha)$ contains the largest element $(1,\alpha,1)$ and consequently $H_q(\Phi(\alpha)) \cong H_q(\mathrm{pt})$ for all $q \geqslant 0$.

If $\alpha \notin \bigcup\limits_{v\in V} M(E_v)$, then $x \neq 1$ or $y \neq 1$ holds for each $(x,\beta,y) \in \Phi(\alpha)$. Therefore $(x,\beta,y) \in L(\alpha)\cup R(\alpha)$. It follows that $\Phi(\alpha) = L(\alpha)\cup R(\alpha)$.

Now we shall prove the acyclicity of $L(\alpha)$, $R(\alpha)$, and $L(\alpha)\cap R(\alpha)$. Then, by Lemma 1.17, we shall obtain the acyclicity of $L(\alpha)\cup R(\alpha)$. If $\alpha = f\circ g$, then denote g by $f^{-1}\alpha$ and f by αg^{-1}.

- $L(\alpha) = \bigcup\limits_{i=1}^{m} L_i(\alpha)$, $L_{i_1}(\alpha)\cap\cdots\cap L_{i_k}(\alpha) = (a_{i_1}\cdots a_{i_k})_*\Phi((a_{i_1}\cdots a_{i_k})^{-1}\alpha)$. Since the map $(a_{i_1}\cdots a_{i_k})_*$ makes an isomorphism of the partially ordered set $\Phi((a_{i_1}\cdots a_{i_k})^{-1}\alpha)$ with a closed subset, the intersections $L_{i_1}(\alpha)\cap\cdots\cap L_{i_k}(\alpha)$ are acyclic. By Corollary 1.18, we obtain the acyclicity of $L(\alpha)$.

- The acyclicity of $R(\alpha) = \bigcup\limits_{j=1}^{n} R_j(\alpha)$ is proved similarly.

- $L(\alpha) \cap R(\alpha) = \bigcup_{i=1}^{m} (L_i(\alpha) \cap R(\alpha))$, $L_{i_1}(\alpha) \cap \cdots \cap L_{i_k}(\alpha) \cap R(\alpha) = \{(a_{i_1} \cdots a_{i_k} x, \beta, y) : \beta \in \bigcup_{v \in V} M(E_v), y \neq 1, a_{i_1} \cdots a_{i_k} x \beta y = \alpha\}$. It follows that

$$L_{i_1}(\alpha) \cap \cdots \cap L_{i_k}(\alpha) \cap R(\alpha) = (a_{i_1} \cdots a_{i_k})_* R((a_{i_1} \cdots a_{i_k})^{-1}\alpha)$$

  is isomorphic to the partially ordered set $R((a_{i_1} \cdots a_{i_k})^{-1}\alpha)$ whose acyclicity is proved in the second item.

It follows from Lemma 1.17 that $\Phi(\alpha)$ is acyclic. Q.E.D.

3 Dimension theory of free partially commutative monoids and its applications

In this section, we compute the Leech dimension of any free partially commutative monoid show that it is equal to the homological dimension. We prove that the homological dimension of the augmented category of states is not greater than the supremum of numbers of mutually independent events.

3.1 Leech dimension

Cohomology of monoids M with coefficients in natural systems was first studied by Leech [12], and we thus introduce the following concept:

Definition 3.1. Let M be a monoid. The *Leech dimension* $\operatorname{Ld} M$ is defined by $\operatorname{Ld} M = \operatorname{hd}(\mathfrak{F}M)^{\mathrm{op}}$.

Let $\mathbb{N}$ be the free monoid generated by one element and $\mathbb{N}^n$ its n-fold power for any integer $n \geqslant 1$.

Lemma 3.2. $\operatorname{Ld} \mathbb{N}^n = n$.

Proof. Let $\mathscr{C}$ be a small category and $\operatorname{cd} \mathscr{C} = \sup\{n : \varprojlim^n_{\mathscr{C}} \neq 0\}$ its *cohomological dimension*. It easy to see that $\operatorname{hd} \mathscr{C}^{\mathrm{op}} \leqslant \operatorname{cd} \mathscr{C}$. If $\mathscr{C}$ is a free category, then $\operatorname{cd} \mathfrak{F}\mathscr{C} \leqslant 1$ by [1]. Hence $\operatorname{Ld} M \leqslant 1$ for any free monoid M. It follows that $\operatorname{hd}(\mathfrak{F}\mathbb{N}^n)^{\mathrm{op}} \leqslant n$ by Lemma 1.15. Since $H_n((\mathfrak{F}\mathbb{N}^n)^{\mathrm{op}} \cong H_n(\mathbb{N}^n)$ by Corollary 1.10 and $H_n(\mathbb{N}^n) \cong H_1(\mathbb{N})^{\otimes n} \cong \mathbb{Z}$ by the Künneth formula, we have $\operatorname{hd}(\mathfrak{F}\mathbb{N}^n)^{\mathrm{op}} = n$. Q.E.D.

Theorem 3.3. Let $M(E, I)$ be a free partially commutative monoid. If the maximal number of mutually independent generators is equal to $n \in \mathbb{N}$, then $\operatorname{Ld} M(E, I) = n$.

Proof. Let $E = \bigcup_{v \in V} E_v$ where E_v are the maximal subsets of mutually independent generators and $M(E_v) \subseteq M(E, I)$ submonoids generated by E_v.

Since $\bigcup_{v\in V} \mathfrak{F}M(E_v) \subseteq \mathfrak{F}M(E,I)$ is a closed subcategory, Lemma 1.14 implies the inequality

$$\mathrm{hd}(\bigcup_{v\in V} \mathfrak{F}M(E_v))^{\mathrm{op}} \leqslant \mathrm{hd}(\mathfrak{F}M(E,I))^{\mathrm{op}}.$$

The inclusion $\bigcup_{v\in V} \mathfrak{F}M(E_v) \subseteq \mathfrak{F}M(E,I)$ is strongly coinitial by Lemma 2.3. We obtain by Corollary 1.12 that

$$\mathrm{hd}(\bigcup_{v\in V} \mathfrak{F}M(E_v))^{\mathrm{op}} \geqslant \mathrm{hd}(\mathfrak{F}M(E,I))^{\mathrm{op}}.$$

Therefore $\mathrm{hd}(\bigcup_{v\in V} \mathfrak{F}M(E_v))^{\mathrm{op}} = \mathrm{hd}(\mathfrak{F}M(E,I))^{\mathrm{op}}$. It now follows from Lemma 1.14 that $\mathrm{hd}(\mathfrak{F}M(E,I))^{\mathrm{op}} = \sup_{v\in V} \mathrm{hd}(\mathfrak{F}M(E_v))^{\mathrm{op}})$, and thus $\mathrm{Ld}\, M(E,I) = \sup_{v\in V} \mathrm{Ld}\, M(E_v)$. If the maximum of cardinalities of E_v is equal to $n \in \mathbb{N}$, then it follows from Lemma 3.2 that $\mathrm{Ld}\, M(E,I) = n$. Q.E.D.

3.2 Homological dimension of a free partially commutative monoid

We confirm the conjecture from [6] about the homological dimension of a free partially commutative monoid.

Theorem 3.4. Let $M(E,I)$ be a free partially commutative monoid. If the maximal number of its mutually independent generators is equal to $n < \infty$, then $\mathrm{hd}\, M(E,I) = n$.

Proof. By Lemma 1.9, the functor $t : \mathfrak{F}(M(E,I)^{\mathrm{op}}) \to M(E,I)^{\mathrm{op}}$ is strongly coinitial. Since $\mathfrak{F}(M(E,I)^{\mathrm{op}})$ and $\mathfrak{F}M(E,I)$ are isomorphic, there is a strongly coinitial functor $\mathfrak{F}M(E,I) \to M(E,I)^{\mathrm{op}}$. Hence $\mathrm{hd}(\mathfrak{F}M(E,I))^{\mathrm{op}} \geqslant \mathrm{hd}\, M(E,I)$. It follows from Theorem 3.3 that $\mathrm{hd}\, M(E,I) \leqslant n$. Prove that equality is true. Take a maximal subset E_v of mutually independent generators which consists of n elements. Since $M(E_v) \cong \mathbb{N}^n$, we can obtain by Künneth formula that $H_n(M(E_v)) \cong \mathbb{Z}$. Consequently, by Corollary 1.10, $H_n(\mathfrak{F}M(E_v)^{\mathrm{op}}) \cong \mathbb{Z}$. Lemma 1.13 gives an isomorphism

$$\underrightarrow{\lim}_n^{\mathfrak{F}M(E,I)^{\mathrm{op}}} (\Delta\, \mathbb{Z})_{\mathfrak{F}M(E_v)} \cong \mathbb{Z}\ ,$$

where $(\Delta\, \mathbb{Z})_{\mathfrak{F}M(E_v)} : \mathfrak{F}M(E,I)^{\mathrm{op}} \to \mathrm{Ab}$ is obtained from $\Delta_{\mathfrak{F}M(E_v)^{\mathrm{op}}}\, \mathbb{Z}$ by adding zeros. The application of $\underrightarrow{\lim}$ to the exact sequence

$$0 \to (\Delta\, \mathbb{Z})_{\mathfrak{F}M(E_v)} \to \Delta\, \mathbb{Z} \to \Delta\, \mathbb{Z}/(\Delta\, \mathbb{Z})_{\mathfrak{F}M(E_v)} \to 0$$

gives a long exact sequence and a monomorphism $\mathbb{Z} \to \varinjlim_n^{\mathfrak{F}M(E,I)^{\mathrm{op}}} \Delta\,\mathbb{Z}$. Therefore

$$\varinjlim_n^{M(E,I)} \Delta\,\mathbb{Z} \cong \varinjlim_n^{\mathfrak{F}M(E,I)^{\mathrm{op}}} \Delta\,\mathbb{Z} \neq 0$$

and consequently $\operatorname{hd} M(E,I) \geqslant n$. Q.E.D.

Remark 3.5. Let $\operatorname{cd} M$ denote the cohomological dimension of a monoid M. On the assumption of Theorem 3.4 we can prove $\operatorname{cd} M(E,I) = n$. It follows that the $M(E,I)$-module $\Delta_{M(E,I)}\,\mathbb{Z}$ has the projective resulutions of length n. Using cubical homology theory of semicubical sets [10] and Theorem 2.3, we can find among them a resolution which generalizes the main result of [16].

3.3 Homological dimension of an asynchronous transition system

An asynchronous transition system is a model of a computational process consisting of several threads which are executed concurrently and contain possibly independent actions. The behaviour of such a process is described by the category of states of its asynchronous transition system. Here we continue our research of the category of states, started in [5, 7]. In [11], we obtained an application for a decomposability condition into a parallel product.

We proved in [5] that any asynchronous transition system may be considered as a pair $T = (M(E,I), X)$ consisting of a free partially commutative monoid $M(E,I)$ and a right pointed set X over $M(E,I)$. We now consider a category $K_*(T)$ whose objects are elements $x \in X$ and morphisms $x \to x'$ triples (μ, x, x') consisting of $\mu \in M(E,I)$, $x \in X$, $x' \in X$ satisfying $x \cdot \mu = x'$.

For any $n \geqslant 0$, the *n-th homology group $H_n(K_*(T), F)$ of an asynchronous transition system T with coefficients in a functor $F : K_*(T) \to \mathrm{Ab}$* is defined as the abelian group $\varinjlim_n^{K_*(T)} F$. Elements of E are called *events*. A subset $\{e_j : j \in J\} \subseteq E$ consists of *mutually independent events* if $(e_j, e_{j'}) \in I$ for any distinct $j, j' \in J$.

Theorem 3.6. If $n > 0$ is the maximal number of mutually independent events of an asynchronous transition system T, then $H_k(K_*(T), F) = 0$ for all $k > n$.

Proof. By Lemma 1.9, the functor $\mathfrak{F}M(E,I) \xrightarrow{t} M(E,I)$ is strongly coinitial. It follows from Corollary 1.12 that $\operatorname{hd}(\mathfrak{F}M(E,I))^{\mathrm{op}} \geqslant \operatorname{hd} M(E,I)^{\mathrm{op}}$. Therefore $\operatorname{hd} M(E,I)^{\mathrm{op}} \leqslant n$. For any diagram $F : K_*(M(E,I), S_*) \to \mathrm{Ab}$ there is a right $M(E,I)$-module $\widetilde{F}$ such that

$$\varinjlim_k^{K_*(M(E,I),S_*)} F \cong \varinjlim_k^{M(E,I)^{\mathrm{op}}} \widetilde{F}$$

by [5, Theorem 5.3]. Hence $H_k(K_*(T), F) = 0$ for all $k > n$. Q.E.D.

4 Concluding remarks

Since the functor $t : \mathfrak{F}M \to M$ is strongly coinitial by Lemma 1.9, the homology groups of any right M-module may be considered as its Leech homology groups. We have seen that the Leech dimension of free partially commutative monoids is easier to analyze than their homological dimension. This allows us to get answers which can not be obtained by a direct approach. A complex for computing the Leech homology of a free partially commutative monoid may be constructed similarly. In some finiteness conditions, it gives algorithms for computing the homology groups of an asynchronous transition system with coefficients in $\mathbb{Z}$.

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