On the cooperation algebra of the connective Adams summand

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Abstract

The aim of this paper is to gain explicit information about the multiplicative structure of $\ell_*\ell$, where ℓ is the connective Adams summand at an odd prime p. Our approach differs from Kane's or Lellmann's because our main technical tool is the MU-based Künneth spectral sequence. We prove that the algebra structure on $\ell_*\ell$ is inherited from the multiplication on a Koszul resolution of ℓ_*BP.

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1 Introduction

Our goal in this paper is to shed light on the structure, in particular on the multiplicative structure, of $\ell_*\ell$, where we work at an odd prime p and ℓ is the Adams summand of the p-localization of the connective K-theory spectrum ku. This was investigated by Kane [6] and Lellmann [9] using Brown-Gitler spectra. Our approach is different and exploits the fact that MU is a commutative $\mathbb{S}$-algebra in the sense of Elmendorf, Kriz, Mandell and May [5] and ℓ is an MU-ring spectrum. In fact it is even an MU-algebra and has a unique E_∞-structure [4]. As a calculational tool, we make use of a Künneth spectral sequence (3.2) converging to $\ell_*\ell$, where we work with a concrete Koszul resolution. Our approach bears some similarities to old work of Landweber [8], who worked without the benefit of the modern development of structured ring spectra. The multiplicative structure on the Koszul resolution gives us control over the convergence of the spectral

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sequence and the multiplicative structure of $\ell_*\ell$. In particular, it sheds light on the torsion.

From Kane's work [6] we know that the torsion in $\ell_*\ell$ is detected by the edge homomorphism into the 0-line of the Adams spectral sequence for $\ell_*\ell$. Our analysis of the Künneth spectral sequence gives an explicit description of the p-torsion elements in $\ell_*\ell$ and we determine their image in the dual of the Steenrod-algebra (see § 8).

The outline of the paper is as follows. We recall some basic facts about complex cobordism, MU, in § 2 and describe the Künneth spectral sequence in § 3. Some background on the Bockstein spectral sequence is given in § 4. The multiplicative structure on the E^2-term of this spectral sequence is made precise in § 5 where we introduce the Koszul resolution we shall use later in terms of its multiplicative generators. We study the torsion part in $\ell_*\ell$ and the torsion-free part separately. The investigation of ordinary and L-homology of ℓ in § 6 leads to the identification of the p-torsion in $\ell_*\ell$ with the u-torsion where $\ell_* = \mathbb{Z}_{(p)}[u]$ with u being in degree $2p-2$. In § 7 we show how to exploit the cofibre sequence

$$\ell \xrightarrow{p} \ell \longrightarrow \ell/p$$

to analyse the Künneth spectral sequence and relate the simpler spectral sequence for ℓ/p to that for ℓ. To that end we prove an auxiliary result on connecting homomorphisms in the Künneth spectral sequence, which is analogous to the well-known geometric boundary theorem (see for instance [15, Chapter 2, §3]). We use the fact that the p- and u-torsion is all simple to show that the Künneth spectral sequence for $\ell_*\ell$ collapses at the E^2-term and that there are no extension issues. We summarize our calculation of $\ell_*\ell$ at the end of that section.

In § 8 we use classical tools from the Adams spectral sequence in order to study torsion phenomena in $\ell_*\ell$. We can describe the torsion in $\ell_*\ell$ in terms of familiar elements which are certain coaction-primitives in the $H\mathbb{F}_p$-homology of ℓ.

We summarize our results on the multiplicative structure on $\ell_*\ell$ at the end of § 9, where we establish congruence relations in the zero line of the Künneth spectral sequence and describe the map from the torsion-free part of $\ell_*\ell$ to $\mathbb{Q} \otimes \ell_*\ell$. Taking this together with the explicit formulae of the multiplication in the torsion part in $\ell_*\ell$ gives a rather comprehensive, though not complete, description of the multiplicative structure of $\ell_*\ell$.

In the appendices we give some results on regular sequences in Hopf algebroids that we find useful in several places in our work, and also an account of the convergence of Massey products in spectral sequences required in our proof of Theorem 7.3.

2 Recollections on MU and ℓ

Throughout, we shall assume all spectra are localized at p for some odd prime p.

Let ku denote connective complex K-theory and let ℓ be the Adams summand, also known as $BP\langle 1\rangle$, so that

$$ku_{(p)} \sim \bigvee_{0\leqslant i\leqslant p-2} \Sigma^{2i}\ell.$$

We have $\ell_* = \pi_*\ell = \mathbb{Z}_{(p)}[u]$ with $u \in \ell_{2(p-1)}$. We shall denote the Adams summand of $KU_{(p)}$ by L; then $L_* = \ell_*[u^{-1}]$.

Let us recall some standard facts for which convenient sources are [1, 17]. Since ℓ is complex oriented,

$$\ell_*MU = \ell_*[m'_n : n \geqslant 1],$$

where $m'_n \in \ell_{2n}MU$ agrees with the m^ℓ_n of Adams [1]. By the Hattori-Stong theorem, the Hurewicz homomorphism $MU_* \longrightarrow \ell_*MU$ is a split monomorphism, so we shall view MU_* as a subring of ℓ_*MU. Now

$$MU_* = \mathbb{Z}_{(p)}[x_n : n \geqslant 1],$$

where $x_n \in MU_{2n}$ and using Milnor's criterion for polynomial generators of MU_* we can arrange that

$$x_n \equiv \begin{cases} pm'_{p^k-1} \text{ mod decomposables} & \text{if } n = p^k - 1 \text{ for some } k, \\ m'_n \quad \text{mod decomposables} & \text{otherwise.} \end{cases}$$

In fact, we can take $x_{p^k-1} = v_k$ to be the Hazewinkel generator which lies in $BP_* \subseteq MU_*$. The following formula recursively determines the Hurewicz image of v_k in $H_*MU = \mathbb{Z}_{(p)}[m_k : k \geqslant 1]$:

$$v_k = pm_{p^k-1} - \sum_{1\leqslant j\leqslant k-1} m_{p^j-1} v_{k-j}^{p^j}. \tag{2.1}$$

In H_*BP with $\lambda_k = m_{p^k-1}$, this corresponds to the familiar formula

$$v_k = p\lambda_k - \sum_{1\leqslant j\leqslant k-1} \lambda_j v_{k-j}^{p^j}. \tag{2.2}$$

We note that

$$\ell_*MU/(\underline{\ell}(x_n) : n \neq p^k - 1 \text{ for any } k) = \ell_*[t_k : k \geqslant 1] = \ell_*BP, \tag{2.3}$$

where $t_k \in \ell_{2p^k-2}BP$ is the image of the standard polynomial generator $t_k \in BP_*BP$ of [1].

Now recall that the natural complex orientation of ℓ factors as

$$\sigma\colon MU \longrightarrow BP \longrightarrow \ell$$

and we can choose the generators x_n so that

$$\sigma_*(x_n) = \begin{cases} u & \text{if } n = p-1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the kernel of the map $BP_* \longrightarrow \ell_*$ is the ideal generated by the Hazewinkel generators $v_2, v_3, \ldots$.

We can also find useful expressions for Hurewicz images $\underline{\ell}(v_n)$ of the v_n in ℓ_*BP and ℓ_*MU. Using standard formulae for the right unit $\eta_R\colon BP_* \longrightarrow BP_*BP$ which can be found in [17], we have for $n \geqslant 2$,

$$\underline{\ell}(v_n) = pt_n + ut_{n-1}^p - u^{p^{n-1}}t_{n-1} + ps'_n + us''_n, \tag{2.4}$$

where $s'_n \in \mathbb{Z}_{(p)}[u, t_1, \ldots, t_{n-1}]$ and $s''_n \in \mathbb{Z}_{(p)}[u, t_1, \ldots, t_{n-2}]$. We also have $\underline{\ell}(v_1) = pt_1 + u$.

We now make some useful deductions.

Proposition 2.1. In the ring $\mathbb{Q} \otimes \ell_*BP$, the sequence

$$\underline{\ell}(v_2), \underline{\ell}(v_3), \ldots, \underline{\ell}(v_n), \ldots$$

is regular and

$$\mathbb{Q} \otimes \ell_*BP/(\underline{\ell}(v_n) : n \geqslant 2) = \mathbb{Q} \otimes \ell_*[t_1] = \mathbb{Q} \otimes \ell_*[v_1].$$

Proof. For each $n \geqslant 1$, pt_n is a polynomial generator for $\mathbb{Q} \otimes \ell_*BP = \mathbb{Q} \otimes \ell_*[t_i : i \geqslant 1]$ over $\mathbb{Q} \otimes \ell_*$. For an alternative approach to this, see Remark A.3. Q.E.D.

Proposition 2.2. In the ring L_*BP, the sequence

$$\underline{\ell}(v_2), \underline{\ell}(v_3), \ldots, \underline{\ell}(v_n), \ldots$$

is regular and

$$\begin{aligned} &L_*BP/(\underline{\ell}(v_n) : n \geqslant 2) = \\ &\quad L_*[t_k : k \geqslant 1]/(t_n^p - u^{p^n-1}t_n + pu^{-1}s'_{n+1} + s''_{n+1} + pu^{-1}t_{n+1} : n \geqslant 1). \end{aligned}$$

In the ring $L_*BP/(p)$, the sequence

$$\underline{\ell}(v_2), \underline{\ell}(v_3), \dots, \underline{\ell}(v_n), \dots$$

is regular and

$$L_*BP/(p, \underline{\ell}(v_n) : n \geqslant 2) = L_*/(p)[t_k : k \geqslant 1]/(t_n^p - u^{p^n-1}t_n + s''_{n+1} : n \geqslant 1).$$

Proof. These results follow from Theorem A.1 and Corollary A.2. Q.E.D.

3 A Künneth spectral sequence for $\ell_*\ell$

We shall describe a calculation of $\ell_*\ell = \pi_*(\ell \wedge \ell)$ that makes use of the Künneth spectral sequence of [5] for MU-modules. This is different from the approach taken by Kane [6], and we feel it offers some insight into the form of answer, especially with regard to multiplicative structure.

For any MU-module spectrum F and any spectrum E there is a Künneth (or universal coefficient) spectral sequence [5, IV.4.5]

$$\begin{aligned}\mathrm{E}^2_{s,t} &= \mathrm{Tor}^{MU_*}_{s,t}(\pi_*(E \wedge MU), \pi_*F) \\ &\Longrightarrow \pi_*((E \wedge MU) \wedge_{MU} F) \cong \pi_*(E \wedge F) = E_*F. \qquad (3.1)\end{aligned}$$

Note that in certain cases this spectral sequence is actually multiplicative ([3, Lemma 1.3], see also Appendix B); in particular for $E = F = \ell$ we obtain a multiplicative spectral sequence

$$\mathrm{E}^2_{s,t} = \mathrm{Tor}^{MU_*}_{s,t}(\pi_*(\ell \wedge MU), \pi_*\ell) \Longrightarrow \ell_*\ell. \qquad (3.2)$$

Now consider the MU_*-module ℓ_*. We can assume that the complex orientation gives rise to a ring isomorphism

$$MU_*/(x_n : n \neq p-1) \xrightarrow{\cong} \ell_*.$$

There is a Koszul resolution of ℓ_* as a module over MU_*,

$$\Lambda_{MU_*}(e_r : 0 < r \neq p-1) \longrightarrow \ell_* \to 0,$$

where $\Lambda_{MU_*}(e_r : 0 < r \neq p-1)$ is the exterior algebra generated by elements e_r of bidegree $(1, 2r)$ whose differential d is the derivation which satisfies $d(e_r) = x_r$.

For arbitrary E and $F = \ell$, the E^2-term of the spectral sequence (3.1) is the homology of the complex

$$E_*MU \otimes_{MU_*} \Lambda_{MU_*}(e_r : 0 < r \neq p-1) \cong \Lambda_{E_*MU}(e_r : 0 < r \neq p-1)$$

with differential $\mathrm{id} \otimes d$ which corresponds to the differential d taking values in the latter complex. From (2.3) we find that the homology of this complex is

$$\mathrm{H}_*(\Lambda_{E_*MU}(e_r : 0 < r \neq p-1), d) = \mathrm{H}_*(\Lambda_{E_*BP}(\varepsilon_r : r \geqslant 2), d), \tag{3.3}$$

where ε_r has bidegree $(1, 2p^r - 2)$ and $d(\varepsilon_r) = v_r$.

Proposition 3.1. Suppose that the E-theory Hurewicz images $\underline{e}(v_k)$ with $k \geqslant 2$ form a regular sequence in E_*BP. Then the complex

$$\Lambda_{E_*BP}(\varepsilon_r : r \geqslant 2) \longrightarrow E_*BP/(\underline{e}(v_r) : r \geqslant 2) \to 0$$

is acyclic and

$$\mathrm{Tor}^{MU_*}_{s,*}(E_*MU, \ell_*) = \begin{cases} E_*BP/(\underline{e}(v_r) : r \geqslant 2) & \text{if } s = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Therefore the Künneth spectral sequence of (3.1) degenerates to give an isomorphism

$$E_*BP/(\underline{e}(v_r) : r \geqslant 2) \xrightarrow{\cong} E_*\ell.$$

The regularity condition of this result applies for each of the cases $E = \ell\mathbb{Q}$, L/p by Propositions 2.2 and 2.1. We do not have a proof that it holds for the case $E = L$, however the following provides a substitute.

Proposition 3.2. Suppose that E is a p-local Landweber exact spectrum. Then the complex

$$\Lambda_{E_*BP}(\varepsilon_r : r \geqslant 2) \longrightarrow E_*BP/(\underline{e}(v_r) : r \geqslant 2) \to 0$$

is acyclic and the conclusion of Proposition 3.1 is valid.

Proof. There are isomorphisms of complexes

$$\begin{aligned} E_*MU \otimes_{MU_*} \Lambda_{MU_*}(e_r : 0 < r \neq p-1) \\ &\cong E_* \otimes_{MU_*} MU_*MU \otimes_{MU_*} \Lambda_{MU_*}(e_r : 0 < r \neq p-1) \\ &\cong E_* \otimes_{MU_*} \Lambda_{MU_*MU}(e_r : 0 < r \neq p-1) \\ &\cong E_* \otimes_{MU_*} \Lambda_{MU_*BP}(\varepsilon_r : r \geqslant 2). \end{aligned}$$

The sequence $v_2, v_3, \ldots$ is regular in MU_*, so $\underline{mu}(v_2), \underline{mu}(v_3), \ldots$ is also regular in MU_*BP, by Theorem A.1. Therefore

$$\Lambda_{MU_*BP}(\varepsilon_r : r \geqslant 2) \longrightarrow MU_*BP/(\underline{mu}(v_r) : r \geqslant 2) \to 0$$

is an exact complex of MU_*BP-modules. The differentials in the complex $\Lambda_{MU_*BP}(\varepsilon_r : r \geqslant 2)$ are coproducts of multiplication by $\underline{mu}(v_r)$ on $MU_*BP = MU_*MU \otimes_{MU_*} BP_*$, and these are all MU_*MU-comodule morphisms by Theorem A.1(i). The hypothesis on E means that the functor $E_* \otimes_{MU_*} (-)$ is exact on the category of left MU_*MU-comodules, hence the complex

$$E_* \otimes_{MU_*} \Lambda_{MU_*BP}(\varepsilon_r : r \geqslant 2) \longrightarrow E_* \otimes_{MU_*} MU_*BP/(\underline{mu}(v_r) : r \geqslant 2) \to 0,$$

is exact. From this we obtain the result. Q.E.D.

Of course, this result applies when $E = L$. Later we shall also consider some cases where these regularity conditions do not hold.

4 Bockstein spectral sequences

We follow [16, p. 158] in this account. Let R be a graded commutative ring and suppose that we have an exact couple of graded R-modules

$$\begin{tikzcd} A^0_* \arrow[rr, "x\cdot"] & & A^0_* \arrow[dl, "j^0"] \\ & B^0_* \arrow[ul, "\delta^0"] & \end{tikzcd}$$

where δ^0 is a map of degree $-|x| - 1$ and $x\cdot$ is multiplication by $x \in R$. Then there are inductively defined exact couples

$$\begin{tikzcd} A^r_* \arrow[rr, "x\cdot"] & & A^r_* \arrow[dl, "j^r"] \\ & B^r_* \arrow[ul, "\delta^r"] & \end{tikzcd}$$

and an associated spectral sequence (B^r, d^r) with $B^{r+1}_* = \mathrm{H}(B^r_*, d^r)$. For each $r \geqslant 1$, there are exact sequences

$$0 \to A^0_n/(xA^0_{n-|x|} + {}_{x^r}A^0_n) \xrightarrow{\bar{j}^r} B^r_n \xrightarrow{\delta^r} {}_xA^0_{n-|x|-1} \cap x^r A^0_{n-|x|-1-r|x|} \to 0, \tag{4.1}$$

where

$${}_{x^r}A^0_n = \ker(x^r \colon A^0_n \longrightarrow A^0_{n+r|x|}), \quad {}_{x^\infty}A^0_n = \bigcup_{r\geqslant 1} {}_{x^r}A^0_n.$$

In particular, if $B^1_n = B^\infty_n = 0$ for some n, we obtain the following:

$${}_{x^\infty}A_n = {}_xA_n, \tag{4.2}$$

$$\ker \delta^0 = \ker d^0 = \operatorname{im} j^0. \tag{4.3}$$

Let $\bar{\ell}$ denote the cofibre of the multiplication by p in the sequence

$$\ell \xrightarrow{p} \ell \xrightarrow{\varrho} \bar{\ell} = \ell/p.$$

We shall make use of the following special case of this situation in our proof of Theorem 7.3. The reader is referred to § 8 for more on the ordinary homology of ℓ.

Proposition 4.1. All u-torsion in $\bar{\ell}_*\ell$ is simple.

Proof. We make use of a Bockstein spectral sequence as above. Setting $A^0_* = \bar{\ell}_*\ell$ and $B^0_* = H_*(\ell;\mathbb{F}_p)$ (where $x = u$ acts trivially), the differential is essentially the Milnor operation Q^1 acting on

$$H_*(\ell;\mathbb{F}_p) = \mathbb{F}_p[\zeta_1,\zeta_2,\ldots] \otimes \Lambda(\bar{\tau}_2,\bar{\tau}_3,\ldots)$$

by

$$Q^1(\bar{\tau}_n) = \zeta^p_{n-1}.$$

Hence we have

$$B^\infty_* = B^1_* = \mathbb{F}_p[\zeta_1,\zeta_2,\ldots]/(\zeta^p_1,\zeta^p_2,\ldots).$$

The composition $BP_*BP \longrightarrow \bar{\ell}_*\ell \longrightarrow H_*(\ell;\mathbb{F}_p)$ maps t_i to ζ_i. As u does not annihilate t_i the maps $\bar{j}^r$ for all $r \geqslant 1$ are surjective. In particular, from (4.1) the u-torsion in $\bar{\ell}_*\ell$ intersected with the multiples of u is trivial. Q.E.D.

5 Generalized Koszul complexes and Bockstein spectral sequences

Let R be a commutative ring and $x \in R$ a non-zero divisor which is also not a unit. Let $w_1, w_2, w_3, \ldots$ be a (possibly finite) regular sequence in R which reduces to a regular sequence in $R/(x)$.

The Koszul complex $(\Lambda_R(e_r : r \geqslant 1), d)$ whose differential is the R-derivation determined by $d(e_r) = w_r$ provides a resolution

$$\Lambda_R(e_r : r \geqslant 1) \longrightarrow R/(w_r : r \geqslant 1) \to 0$$

of $R/(w_r : r \geqslant 1)$ by R-modules.

Now consider the sequence $xw_1, xw_2, xw_3, \ldots$ which is not regular in R since for $s > r$,

$$w_r(xw_s) = w_s(xw_r).$$

The Koszul complex $(\Lambda_R(e'_r : r \geqslant 1), d')$ with differential satisfying $d'(e'_r) = xw_r$ is no longer exact but does augment onto $R/(xw_r : r \geqslant 1)$. Notice that there is a monomorphism of differential graded R-algebras

$$j\colon \Lambda_R(e'_r : r \geqslant 1) \longrightarrow \Lambda_R(e_r : r \geqslant 1); \quad j(e'_r) = xe_r,$$

and this covers the reduction map $R/(xw_r : r \geqslant 1) \longrightarrow R/(w_r : r \geqslant 1)$. Using this, we shall view $\Lambda_R(e'_r : r \geqslant 1)$ as a subcomplex of $\Lambda_R(e_r : r \geqslant 1)$. We want to determine the homology of $(\Lambda_R(e'_r : r \geqslant 1), d')$.

Suppose that $z \in \Lambda_R(e'_r : r \geqslant 1)_n$ with $n > 0$ and $d'(z) = 0$. Then working in $\Lambda_R(e_r : r \geqslant 1)$ we have $d(j(z)) = 0$, so by exactness of the latter complex, there is an element

$$y = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_{n+1}} y_{i_1,i_2,\ldots,i_{n+1}} e_{i_1} e_{i_2} \cdots e_{i_{n+1}} \in \Lambda_R(e_r : r \geqslant 1)_{n+1}$$

for which $d(y) = j(z)$. But

$$d(y) = \sum_{\substack{1 \leqslant i_1 < i_2 < \cdots < i_{n+1} \\ 1 \leqslant k \leqslant n+1}} (-1)^k w_{i_k} y_{i_1,i_2,\ldots,i_{n+1}} e_{i_1} e_{i_2} \cdots \widehat{e}_{i_k} \cdots e_{i_{n+1}}.$$

Since we have

$$j(z) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_n} x^n z_{i_1,i_2,\ldots,i_n} e_{i_1} e_{i_2} \cdots e_{i_n},$$

using the regularity assumption we find that each $y_{i_1,i_2,\ldots,i_{n+1}}$ has the form

$$y_{i_1,i_2,\ldots,i_{n+1}} = x^n y'_{i_1,i_2,\ldots,i_{n+1}}$$

for some $y'_{i_1,i_2,\ldots,i_{n+1}} \in R$ and therefore

$$z = \sum_{\substack{1 \leqslant i_1 < i_2 < \cdots < i_{n+1} \\ 1 \leqslant k \leqslant n+1}} (-1)^k w_{i_k} y'_{i_1,i_2,\ldots,i_{n+1}} e'_{i_1} e'_{i_2} \cdots \widehat{e'}_{i_k} \cdots e'_{i_{n+1}}.$$

Notice that

$$xz = d'\left(\sum_{1 \leqslant i_1 < i_2 < \cdots < i_{n+1}} y'_{i_1,i_2,\ldots,i_{n+1}} e'_{i_1} e'_{i_2} \cdots e'_{i_{n+1}} \right).$$

Therefore x annihilates the n-th homology of $\Lambda_R(e'_r : r \geqslant 1)$ for $n > 0$, and hence it is an $R/(x)$-module spanned by the elements

$$\Delta_x(i_1, i_2, \ldots, i_{n+1}) = \sum_{1 \leqslant k \leqslant n+1} (-1)^k w_{i_k} e'_{i_1} e'_{i_2} \cdots \widehat{e'}_{i_k} \cdots e'_{i_{n+1}} \tag{5.1}$$

for collections of distinct integers $i_1, i_2, \ldots, i_{n+1} \geqslant 1$. Clearly, for a permutation $\sigma \in \mathrm{S}_{n+1}$,

$$\Delta_x(i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(n+1)}) = \operatorname{sign} \sigma \, \Delta_x(i_1, i_2, \ldots, i_{n+1}).$$

Thus we shall often restrict attention to indexing sequences satisfying

$$1 \leqslant i_1 < i_2 < \cdots < i_{n+1}.$$

These elements satisfy some further additive and multiplicative relations.

Proposition 5.1. Let $r, s \geqslant 2$ and suppose that $i_1, i_2, \ldots, i_r \geqslant 1$ and $j_1, j_2, \ldots, j_s \geqslant 1$ are sequences of distinct integers. Let

$$t = \#\{i_1, i_2, \ldots, i_r\} \cup \{j_1, j_2, \ldots, j_s\}$$

and write

$$\{k_1, k_2, \ldots, k_t\} = \{i_1, i_2, \ldots, i_r\} \cup \{j_1, j_2, \ldots, j_s\}$$

with $1 \leqslant k_1 < k_2 < \cdots < k_t$. Then the following identities are satisfied in each of $\Lambda_R(e'_r : r \geqslant 1)$ and $\mathrm{H}_*(\Lambda_R(e'_r : r \geqslant 1), d')$:

$$\Delta_x(i_1, i_2, \ldots, i_r)\Delta_x(j_1, j_2, \ldots, j_s) = \\ = \begin{cases} 0 & \text{if } t \leqslant r+s-2, \\ (-1)^a w_{k_m} \Delta_x(k_1, k_2, \ldots, k_t) & \text{if } \left\{\begin{matrix} t = r+s-1 \\ k_m = i_a = j_b \end{matrix}\right\}, \\ \Sigma & \text{if } t = r+s, \end{cases} \tag{5.2a}$$

$$\sum_{j=1}^{r} (-1)^j w_{i_j} \Delta_x(i_1, i_2, \ldots, \widehat{i_j}, \ldots i_r) = 0, \tag{5.2b}$$

where

$$\Sigma := \sum_{j=1}^{r} (-1)^{j+s+1} w_{i_j} \Delta_x(i_1, i_2, \ldots, \widehat{i_j}, \ldots i_r, j_1, j_2, \ldots, j_s).$$

Theorem 5.2. The homology of $(\Lambda_R(e'_r : r \geqslant 1), d')$ is given by

$$\mathrm{H}_n(\Lambda_R(e'_r : r \geqslant 1), d') \\ = \begin{cases} R/(xw_r : r \geqslant 1) & \text{if } n = 0, \\ R/(x)\{\Delta_x(i_1, i_2, \ldots, i_{n+1}) : 1 \leqslant i_1 < i_2 < \cdots < i_{n+1}\} & \text{if } n > 0, \end{cases}$$

where in the second case, the $R/(x)$-module is generated by the elements $\Delta_x(i_1, i_2, \ldots, i_{n+1})$ indicated, subject to relations given in (5.2b).

Proof. Consider the long exact sequence obtained by taking homology of the exact sequence

$$0 \to R \otimes_R \Lambda_R(e'_r : r \geqslant 1) \to R \otimes_R \Lambda_R(e'_r : r \geqslant 1) \\ \to R/(x) \otimes_R \Lambda_R(e'_r : r \geqslant 1) \to 0.$$

The associated exact couple has

$$A^0_* = \mathrm{H}_*(\Lambda_R(e'_r : r \geqslant 1), d'),$$
$$B^0_* = \mathrm{H}_*(\Lambda_{R/(x)}(e'_r : r \geqslant 1), d') = \Lambda_{R/(x)}(e'_r : r \geqslant 1).$$

Making use of the formula $d^0 e'_r = w_r$ we find that

$$B^1_* = R/(x, w_1, w_2, \ldots).$$

As x is not a zero divisor, the maps $\bar{j}^r$ for $r \geqslant 1$ are all surjective and therefore the x-torsion in A^0_* is all simple. Q.E.D.

Notice that the quotient R-module $R/(xw_r : r \geqslant 1)$ has x-torsion, as does the higher homology, at least if the sequence of w_r's has at least two terms.

We end this section with a result on Massey products in the homology determined in Theorem 5.2, and this will used in the proof of Theorem 7.3.

Proposition 5.3. In the algebra $\mathrm{H}_*(\Lambda_R(e'_r : r \geqslant 1), d')$, for a sequence of distinct natural numbers $i, j, k_1, \ldots, k_n$ with $n \geqslant 2$, the Massey product

$$\langle \Delta_x(i, j), x, \Delta_x(k_1, \ldots, k_n) \rangle$$

is defined and contains $\Delta_x(i, j, k_1, \ldots, k_n)$ with indeterminacy

$$\mathrm{H}_a(\Lambda_R(e'_r : r \geqslant 1), d')\, \Delta_x(i, j) + \mathrm{H}_b(\Lambda_R(e'_r : r \geqslant 1), d')\, \Delta_x(k_1, \ldots, k_n)$$

for suitable degrees a, b.

Proof. We follow the usual conventions for defining Massey products, see [7, §5.4] or [12] for details.

We have

$$d'(e'_i e'_j) = x\Delta_x(i, j),$$
$$d'(e'_{k_1} \cdots e'_{k_2}) = x\Delta_x(k_1, \ldots, k_n),$$

hence a representative of the Massey product $\langle \Delta_x(i, j), x, \Delta_x(k_1, \ldots, k_n) \rangle$ is

$$e'_i e'_j \Delta_x(k_1, \ldots, k_n) + \Delta_x(i, j) e'_{k_1} \cdots e'_{k_n}$$
$$= \sum_{r=1}^{n} (-1)^r w_{k_r} e'_i e'_j e'_{k_1} \cdots \widehat{e'}_{k_r} \cdots e'_{k_n} + w_i e'_j e'_{k_1} \cdots e'_{k_n} - w_j e'_i e'_{k_1} \cdots e'_{k_n}$$
$$= \Delta_x(i, j, k_1, \ldots, k_n),$$

as claimed. Q.E.D.

6 Ordinary and L-homology of ℓ

We can compute $H_*\ell$ making use of the spectral sequence $(\mathrm{E}^r_{*,*}(H), d^r)$ obtained from (3.1) by taking $E = H = H\mathbb{Z}_{(p)}$ and $F = \ell$. This can be compared with the spectral sequence $(\mathrm{E}^r_{*,*}(H\mathbb{Q}), d^r)$ for $H\mathbb{Q}_*\ell$ making use of the morphism of spectral sequences

$$\mathrm{E}^r_{*,*}(H) \longrightarrow \mathrm{E}^r_{*,*}(H\mathbb{Q})$$

induced by the natural map $H \longrightarrow H\mathbb{Q}$. We shall also consider the spectral sequence $(\mathrm{E}^r_{*,*}(\bar{H}), d^r)$ associated with $\bar{H} = H\mathbb{F}_p$.

By (2.2), in the polynomial ring $H\mathbb{Q}_*BP = \mathbb{Q}[\lambda_i : i \geqslant 1]$, the sequence $v_2, v_3, \ldots, v_n, \ldots$ is regular. So by Proposition 3.1 we have

$$\mathrm{E}^2_{s,*}(H\mathbb{Q}) = \begin{cases} \mathbb{Q}[\lambda_i : i \geqslant 1]/(v_k : k \geqslant 2) & \text{if } s = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

Hence this spectral sequence collapses at E^2 and we have

$$H\mathbb{Q}_*\ell = \mathbb{Q}[\lambda_1] = \mathbb{Q}[v_1],$$

where $v_1 = p\lambda_1$. The image of λ_n in $H\mathbb{Q}_*\ell$ can be recursively computed with the aid of the following formula derived from (2.2):

$$\lambda_n = \frac{v_1^{p^{n-1}}\lambda_{n-1}}{p}. \tag{6.2}$$

So we have

$$\lambda_n = \frac{v_1^{(p^n-1)/(p-1)}}{p^n} = p^{p^{n-1}+p^{n-2}+\cdots+p+1-n}\,\lambda_1^{(p^n-1)/(p-1)}. \tag{6.3}$$

Notice that for a monomial in the λ_j's in $H\mathbb{Q}_{2m(p-1)}\ell$, we have

$$\lambda_1^{r_1}\cdots\lambda_n^{r_n} = \frac{v_1^m}{p^{r_1+2r_2+\cdots+nr_n}},$$

for which

$$r_1 + 2r_2 + \cdots + nr_n \leqslant r_1 + r_2\frac{p^2-1}{p-1} + \cdots + r_n\frac{p^n-1}{p-1} = m.$$

This calculation shows that the images of the monomials in the λ_j's in $H\mathbb{Q}_{2m(p-1)}\ell$ are contained in the cyclic $\mathbb{Z}_{(p)}$-module generated by $\lambda_1^m = v_1^m/p^m$. Turning to the spectral sequence $\mathrm{E}^r_{*,*}(H)$, we see that

$$\mathrm{E}^2_{0,*}(H) = H_*BP/(v_j : j \geqslant 2)$$

and the natural map

$$H_{2m(p-1)}BP/(v_j : j \geqslant 2) \longrightarrow H\mathbb{Q}_{2m(p-1)}BP/(v_j : j \geqslant 2)$$

has image equal $\mathbb{Z}_{(p)}\lambda_1^m$. In [1], the analogous result for ku was obtained using the Adams spectral sequence.

Proposition 6.1. For $m \geqslant 0$,

$$\mathrm{im}[H_{2m(p-1)}\ell \longrightarrow H\mathbb{Q}_{2m(p-1)}\ell] = \mathbb{Z}_{(p)}\lambda_1^m = \mathbb{Z}_{(p)}\frac{v_1^m}{p^m}.$$

Hence,

$$\mathrm{im}[H_*\ell \longrightarrow H\mathbb{Q}_*\ell] = \mathbb{Z}_{(p)}[\lambda_1] = \mathbb{Z}_{(p)}[v_1/p].$$

The spectral sequence $(\mathrm{E}^r_{*,*}(\bar{H}), d^r)$ is easy to determine. As for all k $v_k = 0$ in $\bar{H}_*BP$, we find that

$$\mathrm{E}^\infty_{*,*}(\bar{H}) = \mathrm{E}^2_{*,*}(\bar{H}) = \Lambda_{\bar{H}_*BP}(\varepsilon_r : r \geqslant 2).$$

Thus we recover the well-known result that

$$\bar{H}_*\ell = \mathbb{F}_p[t_k : k \geqslant 1] \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(\varepsilon_r : r \geqslant 2),$$

where t_k has degree $2p^k - 2$ and ε_r has degree $2p^r - 1$.
From Propositions 2.2 and 3.1 we have

$$\begin{aligned}\mathrm{Tor}^{MU_*}_{*,*}(L_*MU, \ell_*) &= L_*BP/(\underline{\ell}(v_r) : r \geqslant 2),\\ \mathrm{Tor}^{MU_*}_{*,*}(\bar{L}_*MU, \ell_*) &= \bar{L}_*BP/(\underline{\ell}(v_r) : r \geqslant 2),\end{aligned}$$

where $\bar{L} = L/p$ denotes the spectrum L smashed with the mod p Moore spectrum. As a consequence, the Künneth spectral sequences for $L_*\ell$ and $\bar{L}_*\ell$ degenerate to give

$$L_*BP/(\underline{\ell}(v_r) : r \geqslant 2) \cong L_*\ell, \quad \bar{L}_*BP/(\underline{\ell}(v_r) : r \geqslant 2) \cong \bar{L}_*\ell.$$

Since L_*MU is a free $\mathbb{Z}_{(p)}$-module, multiplication by p gives an exact sequence of right MU_*-modules

$$0 \to L_*MU \xrightarrow{p} L_*MU \longrightarrow \bar{L}_*MU \to 0$$

which induces a long exact sequence on the functor $\mathrm{Tor}^{MU_*}_*(\quad, \ell_*)$ and this collapses to the short exact sequence

$$\begin{aligned}0 \to \mathrm{Tor}^{MU_*}_{0,*}(L_*MU, \ell_*) \xrightarrow{p} \mathrm{Tor}^{MU_*}_{0,*}&(L_*MU, \ell_*)\\ &\longrightarrow \mathrm{Tor}^{MU_*}_{0,*}(\bar{L}_*MU, \ell_*) \to 0.\end{aligned}$$

From this we see that there is a short exact sequence

$$0 \to L_*\ell \xrightarrow{p} L_*\ell \longrightarrow \bar{L}_*\ell \to 0.$$

On tensoring with $\mathbb{Q}$ we easily see that $\mathbb{Q} \otimes \ell_*\ell \longrightarrow \mathbb{Q} \otimes L_*\ell$ is a monomorphism. Hence we have

Proposition 6.2. The ring $L_*\ell$ has no p-torsion and the natural map $\ell_*\ell \longrightarrow L_*\ell$ induces an exact sequence

$$0 \to {}_{p^\infty}(\ell_*\ell) \longrightarrow \ell_*\ell \longrightarrow L_*\ell.$$

Corollary 6.3. We have

$${}_{p^\infty}(\ell_*\ell) = {}_{u^\infty}(\ell_*\ell).$$

Proof. Since $\ell_* \longrightarrow L_* = \ell_*[u^{-1}]$ is a localization, we have $L_*\ell = \ell_*\ell[u^{-1}]$ and

$$\ker(\ell_*\ell \longrightarrow L_*\ell) = {}_{u^\infty}(\ell_*\ell),$$

hence ${}_{u^\infty}(\ell_*\ell) = {}_{p^\infty}(\ell_*\ell)$. Q.E.D.

7 Connecting homomorphisms in the Künneth spectral sequence

In order to gain control over the p-torsion in $\mathrm{Tor}^{MU_*}_{*,*}(\ell_*MU, \ell_*)$, we shall exploit the cofibre sequence

$$\ell \xrightarrow{p} \ell \xrightarrow{\varrho} \bar{\ell} \xrightarrow{\delta} \Sigma\ell. \tag{7.1}$$

To this end we shall relate the geometric connecting morphisms of cofibre sequences to morphisms of Künneth spectral sequences. The method of proof we use in this part is analogous to that of the geometric boundary theorem in [15, II.3].

Suppose that R is a commutative $\mathbb{S}$-algebra and let W be a cofibrant R-module which we fix from now on. Then for any R-module Z there is a Künneth spectral sequence with

$$\mathrm{E}^2_{s,t}(Z) = \mathrm{Tor}^{R_*}_{s,t}(Z_*, W_*) \Longrightarrow \pi_*(Z \wedge_R W).$$

Lemma 7.1. Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

be a cofibre sequence of R-modules with $X \simeq \bigvee_{i=1}^m \Sigma^{n_i} R$ and $\pi_* f$ surjective. Then there is a map of Künneth spectral sequences

$$\mathrm{E}^r_{s,t}(Y) \xrightarrow{\psi^r} \mathrm{E}^r_{s-1,t}(\Sigma^{-1}Z) \quad (r \geqslant 2),$$

such that ψ^2 is the connecting homomorphism

$$\mathrm{Tor}^{R_*}_{s,t}(Y_*, W_*) \longrightarrow \mathrm{Tor}^{R_*}_{s-1,t}((\Sigma^{-1}Z)_*, W_*).$$

Proof. Since $\pi_* f$ is surjective, there is a short exact sequence

$$0 \to (\Sigma^{-1} Z)_* \longrightarrow \bigoplus_{i=1}^{m} \Sigma^{n_i} R_* \longrightarrow Y_* \to 0.$$

This induces a long exact sequence of Tor-groups, in which every third term is trivial, because $\bigoplus_{i=1}^{m} \Sigma^{n_i} R_*$ is R_*-free. Therefore we have an isomorphism

$$\mathrm{Tor}^{R_*}_{s,t}(Y_*, W_*) \xrightarrow{\cong} \mathrm{Tor}^{R_*}_{s-1,t}((\Sigma^{-1} Z)_*, W_*).$$

On the level of projective resolutions, we can splice a resolution $P_{\bullet,*}$ for Y_* together with a resolution $Q_{\bullet,*}$ of $(\Sigma^{-1} Z)_*$ to obtain a trivial split resolution for $\bigoplus_{i=1}^{m} \Sigma^{n_i} R_*$. Thus we obtain a map between exact couples and so obtain the desired map of spectral sequences. Q.E.D.

Theorem 7.2. Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

be a cofibre sequence of R-modules with $\pi_* f$ surjective. Then there is an induced map of Künneth spectral sequences

$$\mathrm{E}^r_{s,t}(Y) \xrightarrow{\varphi^r} \mathrm{E}^r_{s-1,t}(\Sigma^{-1} Z) \quad (r \geqslant 2)$$

such that φ^2 is the connecting homomorphism

$$\mathrm{Tor}^{R_*}_{s,t}(Y_*, W_*) \longrightarrow \mathrm{Tor}^{R_*}_{s-1,t}((\Sigma^{-1} Z)_*, W_*).$$

Proof. Choose a map $f' \colon \bigvee_{i=1}^{m} \Sigma^{n_i} R \longrightarrow Y$ with $\pi_* f'$ surjective and consider the cofibre sequence

$$\bigvee_{i=1}^{m} \Sigma^{n_i} R \xrightarrow{f'} Y \xrightarrow{j} \mathrm{cone}(f').$$

By Lemma 7.1 there is a map of Künneth spectral sequences

$$\mathrm{E}^r_{s,t}(Y) \xrightarrow{\psi^r} \mathrm{E}^r_{s-1,t}(\Sigma^{-1} \mathrm{cone}(f')).$$

As $\pi_* f$ is surjective, the composition $g \circ f'$ is trivial and there is a factorization $g = \xi \circ j$.

$$\begin{tikzcd}
 & \mathrm{cone}(f') \arrow[dr, dotted, "\xi"] & & \\
X \arrow[r, "f"] & Y \arrow[u, "j"] \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \\
 & \bigvee_{i=1}^{m} \Sigma^{n_i} R \arrow[u, "f'"] & &
\end{tikzcd}$$

Now we may define φ^r to be $(\Sigma^{-1}\xi)_* \circ \psi^r$. Q.E.D.

For the connective Adams summand ℓ, we shall consider the cofibre sequence

$$\ell \wedge MU \xrightarrow{\varrho} \bar{\ell} \wedge MU \xrightarrow{\delta} \Sigma\ell \wedge MU \xrightarrow{\Sigma p} \Sigma\ell \wedge MU \tag{7.2}$$

obtained from (7.1) by smashing with MU. The reduction map ϱ is surjective in homotopy and therefore we can apply Theorem 7.2 to obtain a map of Künneth spectral sequences

$$\mathrm{E}^r_{s,t}(\bar{\ell} \wedge MU) \xrightarrow{\varphi^r} \mathrm{E}^r_{s-1,t}(\ell \wedge MU) \quad (r \geqslant 2).$$

In particular, this yields a connecting homomorphism

$$\varphi^2\colon \operatorname{Tor}^{MU_*}_{s,t}(\bar{\ell}_*MU, \ell_*) \longrightarrow \operatorname{Tor}^{MU_*}_{s-1,t}(\ell_*MU, \ell_*).$$

The following result is crucial for understanding the Künneth spectral sequence for $\ell_*\ell$.

Theorem 7.3. Each p-torsion element of $\operatorname{Tor}^{MU_*}_{s,*}(\ell_*MU, \ell_*)$ is the image of an element of $\operatorname{Tor}^{MU_*}_{s+1,*}(\bar{\ell}_*MU, \ell_*)$ under the connecting homomorphism φ^2 and is an infinite cycle.

Before giving the proof, we need some preliminaries. We shall apply the ideas of § 5 in the context of the ring $R = \bar{\ell}_*BP$. Consider the sequence $\underline{\bar{\ell}}(v_2), \underline{\bar{\ell}}(v_3), \ldots$ in $\bar{\ell}_*BP$. By (2.4), we have for $n \geqslant 2$,

$$\underline{\bar{\ell}}(v_n) = ut^p_{n-1} - u^{p^{n-1}}t_{n-1} + us''_n$$

where $s''_n \in \mathbb{F}_p[u, t_1, \ldots, t_{n-2}]$; thus for $n \geqslant 1$ we set

$$w_n = t^p_n - u^{p^n-1}t_n + s''_{n+1}, \tag{7.3}$$

so that $\underline{\bar{\ell}}(v_{n+1}) = uw_n$. This gives a sequence $w_1, w_2, \ldots$ in $\bar{\ell}_*BP$. Now to apply Propositions 3.1 and 3.2 in the case $E = \bar{\ell}$, we require a lemma.

Lemma 7.4. The sequence $w_1, w_2, \ldots$ is regular in $\bar{\ell}_*BP$.

Proof. Recall that $\bar{\ell}_*BP = \mathbb{F}_p[u, t_1, t_2, \ldots]$ is a polynomial algebra over $\mathbb{F}_p$ and so it is an integral domain. Thus w_1 is not a zero divisor. Now suppose that for some $n \geqslant 2$, we have established that $w_1, w_2, \ldots, w_{n-1}$ is regular. We shall set

$$A(n) = \mathbb{F}_p[u, t_1, \ldots, t_{n-1}]/(w_1, w_2, \ldots, w_{n-1}).$$

Then

$$\bar{\ell}_*BP/(w_1, w_2, \ldots, w_{n-1}) = A(n)[t_n, t_{n+1}, \ldots],$$

i.e., this is a polynomial ring over $A(n)$. The image of s''_{n+1} in $\bar{\ell}_*BP$ lies in $A(n)$. Now it is clear from (7.3) that w_n cannot be a zero divisor in $A(n)[t_n, t_{n+1}, \ldots]$ since it has highest monomial term t_n^p. Q.E.D.

Finally we can prove our theorem.

Proof of Theorem 7.3. Making use of the long exact sequence on Tor-groups associated with the short exact sequence

$$0 \to \ell_*MU \xrightarrow{p} \ell_*MU \xrightarrow{\varrho_*} \bar{\ell}_*MU \to 0$$

induced from (7.2), the claim about the p-torsion in $\mathrm{Tor}_{*,*}^{MU_*}(\ell_*MU, \ell_*)$ follows.

We shall prove that the elements $\Delta_u(i_1, \ldots, i_m)$ with $i_1, \ldots, i_m$ distinct are infinite cycles in the Künneth spectral sequence for $\bar{\ell}_*\ell$, then it follows that the elements $\varphi^2\Delta_u(i_1, \ldots, i_m)$ must also be a infinite cycles in the spectral sequence for $\ell_*\ell$.

Our proof will show that $\mathrm{E}^2_{s,t} = \mathrm{E}^\infty_{s,t}$ by induction on total degree $s+t$. Clearly this is true in total degree 0. So assume that it holds for total degree less than $n > 0$, say. To establish the inductive step, it suffices to show that each $\Delta_u(i_1, \ldots, i_m)$ with total degree n is an infinite cycle (we only need consider the case where the i_j are distinct, and such elements of lower total degree are already assume to be infinite cycles). If $m = 2, 3$, elements of form $\Delta_u(i_1, i_2)$ or $\Delta_u(i_1, i_2, i_3)$ are infinite cycles for degree reasons. If $m \geqslant 4$, then by Proposition 5.3, we know that $\Delta_u(i_1, \ldots, i_m)$ lives in the Massey product $\langle \Delta_u(i_1, i_2), u, \Delta_u(i_3, \ldots, i_m)\rangle$ whose indeterminacy consists of infinite cycles which are decomposable in the algebra structure. Using the fact that the elements $\Delta_u(i_1, i_2)$, u, and $\Delta_u(i_3, \ldots, i_m)$ represent homotopy classes, together with Proposition 4.1, we may form the analogous Toda bracket in $\bar{\ell}_*\ell = \pi_*(\bar{\ell} \wedge \ell)$, and this must be represented in the spectral sequence by $\langle \Delta_u(i_1, i_2), u, \Delta_u(i_3, \ldots, i_m)\rangle$. This situation is similar to that discussed in [7, Proposition 5.4.5] and its following paragraph on homology spectral sequences. As we shall see in the following discussion, Kochman's crossing condition 5.4.5(d) is a consequence of our inductive assumption, thus $\Delta_u(i_1, \ldots, i_m)$ must be an infinite cycle as is required to verify the inductive step. The reader is referred to Appendix B, where we give an exposition of the ingredients required for this argument.

In order to see that the crossing condition holds, we note that the induction in the proof of Kochman's proposition 5.4.5 for a triple product $\langle x_1, x_2, x_3\rangle$ only involves the terms $X_{0,1}, X_{2,3}$ in the defining system. Crossing differentials that might occur for $\langle \Delta_u(i_1, i_2), u, \Delta_u(i_3, \ldots, i_m)\rangle$ would involve domains corresponding to total degrees equal to those of the elements $\Delta_u(i_1, i_2)u$ and $u\Delta_u(i_3, \ldots, i_m)$. But both of these degrees are less than n, so all such differentials are trivial by our inductive assumption. Q.E.D. (Theorem 7.3)

Corollary 7.5. Since the elements $\varphi^2\Delta_u(i_1,\dots,i_n)$ generate the torsion in the Künneth spectral sequence for $\ell_*\ell$ as a module over the ring ℓ_*BP, this spectral sequence collapses at the E^2-page.

Remark 7.6. To summarize: The calculation of the rational homology of ℓ in (6.1) tells us that the torsion-free part of $\ell_*\ell$ has to have its origin in the zero-line of the Künneth spectral sequence. Theorem 7.3 gives an explicit description of the p-torsion classes in $\mathrm{Tor}^{MU_*}_{*,*}(\ell_*MU,\ell_*)$ as the image of the elements $\Delta_u(i_1,\dots,i_n)$ in $\mathrm{Tor}^{MU_*}_{*,*}(\bar\ell_*MU,\ell_*)$ under the geometric boundary, so the torsion part is imported from the Künneth spectral sequence for $\bar\ell_*\ell$. The Künneth spectral sequence for $\ell_*\ell$ collapses at the E^2-page.

Later, we need to know that there are no extension problems. From Corollary 6.3 we know that the p-torsion and u-torsion in $\ell_*\ell$ agree. We recall a result from [6, Proposition 9.1].

Proposition 7.7. All torsion in $\ell_*\ell$ is simple, *i.e.*, for every torsion-class $x\in\ell_*\ell$ we have $px=0$ which is equivalent to $ux=0$.

Corollary 7.8. The Künneth spectral sequence for $\ell_*\ell$ collapses at the E^2-page and there are no non-trivial extensions.

8 Detecting homotopy in the Adams spectral sequence

In this section we recall some results about the classical Adams spectral sequence for $\ell_*\ell$. We make heavy use of standard facts about Hopf algebras and the Steenrod algebra [14, 13]. In the following we generically write I for identity morphisms, φ for products and actions, ψ for coproducts and coactions, η for units and ε for counits and we use $\bar x$ for the antipode on an element x. Undecorated tensor products are taken over the ground field.

We write $\bar H_*(-)$ for $H_*(-;\mathbb{F}_p)$ and $\mathcal{A}_*$ for the dual Steenrod algebra,

$$\mathcal{A}_* = \mathbb{F}_p[\zeta_n : n\geqslant 1]\otimes\Lambda(\bar\tau_n : n\geqslant 0),$$

where the coaction is given by

$$\psi(\zeta_n)=\sum_{i=0}^{n}\zeta_i\otimes\zeta_{n-i}^{p^i},\quad \psi(\bar\tau_n)=1\otimes\bar\tau_n+\sum_{i=0}^{n}\bar\tau_i\otimes\zeta_{n-i}^{p^i}.$$

The generators $\zeta_n\in\mathcal{A}_{2p^n-2}$ and $\bar\tau_n\in\mathcal{A}_{2p^n-1}$ are related to the Milnor generators ξ_n,τ_n by the antipode:

$$\zeta_n=\chi(\xi_n),\quad \bar\tau_n=\chi(\tau_n).$$

The sub-comodule algebra

$$\mathcal{B}_* = \mathbb{F}_p[\zeta_n : n \geqslant 1] \otimes \Lambda(\bar{\tau}_n : n \geqslant 2)$$

gives rise to a quotient Hopf algebra

$$\mathcal{E}_* = \mathcal{A}_* // \mathcal{B}_* = \Lambda(\alpha, \beta),$$

where α, β are the residue classes of $\bar{\tau}_0, \bar{\tau}_1$ respectively. Then

$$\mathcal{B}_* = \mathcal{A}_* \square_{\mathcal{E}_*} \mathbb{F}_p.$$

Now the natural map $\ell \longrightarrow \bar{H}$ induces an isomorphism

$$\bar{H}_*(\ell) \xrightarrow{\cong} \mathcal{B}_* \subseteq \mathcal{A}_*$$

and there are isomorphisms of $\mathcal{A}_*$-comodule algebras

$$\bar{H}_*(\ell \wedge \ell) \xrightarrow{\cong} \bar{H}_*(\ell) \otimes \bar{H}_*(\ell) \xrightarrow{\cong} \mathcal{B}_* \otimes \mathcal{B}_* \xrightarrow{\cong} \mathcal{A}_* \square_{\mathcal{E}_*} \mathcal{B}_*. \tag{8.1}$$

The E_2-term of the Adams spectral sequence converging to $\pi_*(\ell \wedge \ell) = \ell_* \ell$ has the form

$$\mathrm{E}^2_{s,t} = \operatorname{Cotor}^{\mathcal{A}_*}_{s,t}(\mathbb{F}_p, \bar{H}_*(\ell \wedge \ell)) \cong \operatorname{Cotor}^{\mathcal{A}_*}_{s,t}(\mathbb{F}_p, \mathcal{A}_* \square_{\mathcal{E}_*} \mathcal{B}_*)$$

and so by making use of a standard change of rings result, we have

$$\mathrm{E}^2_{s,t} \cong \operatorname{Cotor}^{\mathcal{E}_*}_{s,t}(\mathbb{F}_p, \mathcal{B}_*). \tag{8.2}$$

Note that by results of [6], the torsion in $\ell_*\ell$ is detected by the edge homomorphism (which is essentially the Hurewicz homomorphism) into the 0-line

$$\mathrm{E}^2_{0,*} \cong \operatorname{Cotor}^{\mathcal{E}_*}_{0,*}(\mathbb{F}_p, \mathcal{B}_*) = \mathbb{F}_p \square_{\mathcal{E}_*} \mathcal{B}_*.$$

The map involved here is obtained by composing the following $\mathcal{A}_*$-comodule algebra homomorphisms and suitably restricting the codomain:

$$\begin{aligned} \pi_*(\ell \wedge \ell) \longrightarrow \bar{H}_*(\ell) \otimes \bar{H}_*(\ell) \xrightarrow{\cong} \mathcal{B}_* \otimes \mathcal{B}_* &\xrightarrow{I \otimes \psi} \mathcal{B}_* \otimes (\mathcal{A}_* \square_{\mathcal{E}_*} \mathcal{B}_*) \\ &\xrightarrow{\varphi \otimes I} \mathcal{A}_* \square_{\mathcal{E}_*} \mathcal{B}_* \longrightarrow \mathcal{E}_* \square_{\mathcal{E}_*} \mathcal{B}_* \xrightarrow{\cong} \mathcal{B}_*. \end{aligned}$$

Here, the second to last map is induced by the natural projection map $\mathcal{A}_* \longrightarrow \mathcal{A}_* // \mathcal{B}_* = \mathcal{E}_*$ and the final isomorphism is the composition

$$\mathcal{E}_* \square_{\mathcal{E}_*} \mathcal{B}_* \xrightarrow{\text{incl}} \mathcal{E}_* \otimes \mathcal{B}_* \xrightarrow{\varepsilon \otimes I} \mathbb{F}_p \otimes \mathcal{B}_* \xrightarrow{\cong} \mathcal{B}_*.$$

A careful check of what the composition does on primitives shows that it can be expressed as

$$\pi_*(\ell \wedge \ell) \longrightarrow \bar{H}_*(\ell \wedge \ell) \xrightarrow{(\nu\wedge \mathrm{id})_*} \bar{H}_*(\ell), \tag{8.3}$$

where $\nu\colon \bar{H} \wedge \ell \longrightarrow \bar{H}$ is the natural pairing. In particular, this implies that the image of the Hurewicz map for $\ell \wedge \ell$ maps monomorphically into $\bar{H}_*(\ell)$.

It will be useful to have an explicit version of the isomorphism

$$\mathbb{F}_p \Box_{\mathcal{A}_*} (\mathcal{B}_* \otimes \mathcal{B}_*) \cong \mathbb{F}_p \Box_{\mathcal{A}_*} (\mathcal{A}_* \Box_{\mathcal{E}_*} \mathcal{B}_*) \cong \mathbb{F}_p \Box_{\mathcal{E}_*} \mathcal{B}_*.$$

This is just

$$\mathbb{F}_p \Box_{\mathcal{E}_*} \mathcal{B}_* \xrightarrow{\mathrm{incl}} \mathbb{F}_p \otimes \mathcal{B}_* \xrightarrow{I \otimes \psi} \mathbb{F}_p \otimes (\mathcal{A}_* \otimes \mathcal{B}_*),$$

whose image is in fact contained in $\mathbb{F}_p \Box_{\mathcal{A}_*} (\mathcal{B}_* \otimes \mathcal{B}_*)$.

Given these results, we can use them to detect elements of $\ell_*\ell$ in $\mathcal{B}_*$, in particular we can detect the torsion this way. To do this, we need to understand $\mathcal{B}_*$ as an $\mathcal{E}_*$-comodule, in particular the non-trivial $\mathcal{E}_*$-parallelograms of the form

$$\begin{tikzcd}
 & & & x \arrow[dll, "-\beta"'] \arrow[d, "\alpha"] \\
 & x'' \arrow[dl, "\alpha"'] & & x' \arrow[dlll, "\beta"] \\
x''' & & &
\end{tikzcd} \tag{8.4}$$

in which the $\mathcal{E}_*$-coaction satisfies

$$\begin{aligned}
\psi(x) &= 1 \otimes x + \alpha \otimes x' - \beta \otimes x'' + \beta\alpha \otimes x''', \\
\psi(x') &= 1 \otimes x' + \beta \otimes x''', \\
\psi(x'') &= 1 \otimes x'' + \alpha \otimes x'''.
\end{aligned}$$

Then x''' is an element of $\mathbb{F}_p \Box_{\mathcal{E}_*} \mathcal{B}_*$ which corresponds to an $H\mathbb{F}_p$ wedge summand in $\ell \wedge \ell$ and a correponding torsion element. Of course, these elements can be expressed in terms of the homology action of Q_0 and Q_1, *i.e.*,

$$x' = Q_0 x, \quad x'' = -Q_1 x, \quad x''' = Q_1 Q_0 x.$$

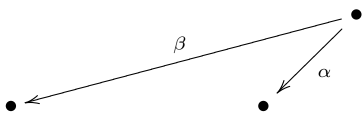
Now Margolis [10, Chapter 18, Theorem 5] dualized to a homology version for $\mathcal{E}_*$-comodules tells us that $\mathcal{B}_*$ uniquely decomposes into a coproduct of comodules isomorphic (up to grading) to $\mathcal{E}_*$, together with a comodule containing no free summand and isomorphic to a coproduct of lightning flash comodules. The latter summand does not concern us for now since all the torsion in $\ell_*\ell$ comes from the $H\mathbb{F}_p$ wedge summands as above corresponding to the free summand. In fact, Adams and Priddy [2, proof of Proposition 3.12] determine the stable type of the lightning flash comodules, in particular, the stable class of the $\mathcal{E}_*$-comodule $\mathcal{B}_*$ is shown to be

$$\bigotimes_{r\geqslant 0}(1 + L_r + L_r^2 + \cdots + L_r^{p-1}), \tag{8.5}$$

where

$$L_r = \Sigma^{a(r)} J^{b(r)}, \quad a(r) + b(r) = 2(p-1)p^r, \quad b(r) = p^{r-1} + \cdots + p + 1.$$

Here $J = \mathcal{E}_*/\mathbb{F}_p$ is the coaugmentation coideal of $\mathcal{E}_*$, represented by the following diagram



and Σ is the trivial comodule $\mathbb{F}_p$ assigned degree 1. Furthermore, all products are tensor products over $\mathbb{F}_p$ taken in the stable comodule category of $\mathcal{E}_*$.

Now the most obvious candidates for the tops of $\mathcal{E}_*$-parallelograms are the elements

$$\bar{\tau}_{i_1}\bar{\tau}_{i_2}\cdots\bar{\tau}_{i_{n+1}} \qquad (1 < i_1 < i_2 < \cdots < i_{n+1}, \quad n \geqslant 1).$$

These can be multiplied by monomials in the ζ_j to obtain others.

Theorem 8.1. Consider the $\mathbb{F}_p$-vector subspace $\mathcal{V} \subseteq \mathbb{F}_p\square_{\mathcal{E}_*}\mathcal{B}_*$ spanned by $\mathbb{F}_p[\zeta_i : i \geqslant 1]$-scalar multiples of the elements 1 and

$$Q_1Q_0(\bar{\tau}_{i_1}\bar{\tau}_{i_2}\cdots\bar{\tau}_{i_{n+1}}) \qquad (1 < i_1 < i_2 < \cdots < i_{n+1}, \quad n \geqslant 1). \tag{8.6}$$

Then $\mathcal{V}$ consists of all the elements in $\mathbb{F}_p\square_{\mathcal{E}_*}\mathcal{B}_*$ which are the images of torsion elements under the composition of the Hurewicz homomorphism $\pi_*(\ell\wedge\ell) \longrightarrow \bar{H}_*(\ell\wedge\ell)$ and the identification of the homology $\bar{H}_*(\ell\wedge\ell)$ with $\mathbb{F}_p\square_{\mathcal{E}_*}\mathcal{B}_*$.

Proof. Clearly $\mathbb{F}_p[\zeta_i : i \geqslant 1] \subseteq \mathbb{F}_p\square_{\mathcal{E}_*}\mathcal{B}_*$. Now we know that the Künneth spectral sequence for $\ell_*\ell$ collapses and there are no additive extension problems. We need to understand the mod p Hurewicz images of elements represented by the elements arising from the $\Delta_u(i_1, \ldots, i_{s+2})$ in

$\mathrm{Tor}^{MU_*}_{s+1,*}(\bar{\ell}_* MU, \ell_*)$, since these will give an additive basis for the p-torsion in $\ell_*\ell$.

$$
\begin{tikzcd}
\mathrm{Tor}^{MU_*}_{s+1,*}(\bar{\ell}_* MU, \ell_*) \arrow[d, "\delta"'] & & \\
\mathrm{Tor}^{MU_*}_{s,*}(\ell_* MU, \ell_*) \arrow[r, Rightarrow] \arrow[d] & \pi_*(\ell \wedge \ell) \arrow[d] \arrow[dr, bend left] & \\
\mathrm{Tor}^{MU_*}_{s,*}((\bar{H} \wedge \ell)_* MU, \ell_*) \arrow[r, Rightarrow] \arrow[d] & \pi_*(\bar{H} \wedge \ell \wedge \ell) \arrow[d] & \mathbb{F}_p \square_{\mathcal{A}_*} \bar{H}_*(\ell \wedge \ell) \arrow[l] \arrow[d] \\
\mathrm{Tor}^{MU_*}_{s,*}(\bar{H}_* MU, \ell_*) \arrow[r, Rightarrow] & \bar{H}_*(\ell) & \mathbb{F}_p \square_{\mathcal{E}_*} \bar{H}_*(\ell) \arrow[l]
\end{tikzcd}
$$

The Künneth spectral sequence (3.1) for $E_*\ell$ is natural for maps of ring spectra $E \longrightarrow F$. Therefore the map (8.3) corresponds in the spectral sequence to the composition of the two vertical maps in the left column in the diagram above. As the Hurewicz homomorphism has its image in the primitives of $\bar{H}_*(\ell \wedge \ell)$, it follows that the elements $\Delta_u(i_1, \ldots, i_{s+2})$ up to a unit map to

$$
\begin{aligned}
&Q_0 Q_1(\bar{\tau}_{i_1} \cdots \bar{\tau}_{i_{s+2}}) = \\
&\qquad \sum_{1<t<r\leqslant s+2} (-1)^{r+t} (\zeta_{i_r} \zeta_{i_t-1}^p - \zeta_{i_t} \zeta_{i_r-1}^p)\, \bar{\tau}_{i_1} \bar{\tau}_{i_2} \cdots \widehat{\bar{\tau}}_{i_t} \cdots \widehat{\bar{\tau}}_{i_r} \cdots \bar{\tau}_{i_{s+2}}.
\end{aligned}
$$

Q.E.D.

Remark 8.2. The torsion in $\pi_*(\ell \wedge \ell)$ maps injectively into $\mathbb{F}_p \square_{\mathcal{A}_*} (\mathcal{B}_* \otimes \mathcal{B}_*)$, which in turn is identified with $\mathbb{F}_p \square_{\mathcal{E}_*} \mathcal{B}_*$, therefore Theorem 8.1 shows that the elements $Q_1 Q_0(\bar{\tau}_{i_1} \bar{\tau}_{i_2} \cdots \bar{\tau}_{i_n})$ with $n \geqslant 3$ correspond to nilpotent elements; only elements of the form $Q_1 Q_0(\bar{\tau}_r \bar{\tau}_s)$ are not nilpotent.

Example 8.3. For every prime p, the first torsion class in $\mathrm{Tor}^{BP_*}_{*,*}(\ell_* BP, \ell_*)$ occurs in degree $2(p^3 + p^2 - p - 1)$ and this class survives to $\ell_*\ell$. The lowest degree element appearing as the bottom of a parallelogram is

$$Q_1 Q_0(\bar{\tau}_2 \bar{\tau}_3) = \zeta_2^{p+1} - \zeta_1^p \zeta_3.$$

The coaction map ψ sends this element to the Hurewicz image of the corresponding torsion element of $\ell_*\ell$ in $\bar{H}_*(\ell \wedge \ell)$.

9 Multiplicative structure of $\ell_*\ell$

In this section we establish congruence relations in the zero line of the Künneth spectral sequence. These are derived in BP_*BP and mapped under the natural map. In fact they are first produced in $\mathbb{Q}\otimes BP_*BP$ then interpreted in the subring BP_*BP.

We describe the map from the torsion-free part of $\ell_*\ell$ to $\ell_*\ell\otimes\mathbb{Q}$ and summarize our results about the multiplicative structure of $\ell_*\ell$ at the end of this section.

It will be useful to have the following straightforward generalization of a well-known result (which corresponds to the case where $t=1$).

Lemma 9.1. Let R be a commutative ring, p a prime and $t\in R$. If $x,y,z\in R$ satisfy $z\equiv px+ty \bmod (pt)$, then for all $k\geqslant 0$,

$$z^{p^k}\equiv p^{p^k}x^{p^k}+t^{p^k}y^{p^k} \bmod (p^{k+1}t).$$

We shall work with the Hazewinkel generators v_n of (2.2). The following standard formula for the right unit $\eta_R\colon \mathbb{Q}\otimes BP_*\longrightarrow \mathbb{Q}\otimes BP_*BP$ can be found in [17, p. 24]:

$$\eta_R(\lambda_n)=\sum_{0\leqslant j\leqslant n}\lambda_j t_{n-j}^{p^j}. \tag{9.1}$$

On combining this with (2.2) we obtain

$$\eta_R(v_n)=\sum_{0\leqslant i\leqslant n}p\lambda_i t_{n-i}^{p^i}-\sum_{\substack{1\leqslant i\leqslant n-1\\0\leqslant j\leqslant i}}\lambda_j t_{i-j}^{p^j}\eta_R(v_{n-i})^{p^i}$$

and hence

$$\begin{gathered}\eta_R(v_n)=\\ \sum_{0\leqslant i\leqslant n}p\lambda_i t_{n-i}^{p^i}-\sum_{0\leqslant i\leqslant n-1}\lambda_i t_{n-1-i}^{p^i}\eta_R(v_1)^{p^{n-1}}-\sum_{\substack{1\leqslant i\leqslant n-2\\0\leqslant j\leqslant i}}\lambda_j t_{i-j}^{p^j}\eta_R(v_{n-i})^{p^i}.\end{gathered} \tag{9.2}$$

Remark 9.2. The left hand side of equation (9.2) lies in $BP_*BP\subseteq\mathbb{Q}\otimes BP_*BP$, therefore so does the right hand side. However, because of the presence of denominators in the terms involving the λ_r, care needs to be exercised when using this equation.

For example, since $cp_r = [\mathbb{C}P^{p^r-1}] = p^r\lambda_r \in BP_*$, we can certainly deduce that in BP_*BP modulo the ideal $(\eta_R(v_2),\ldots,\eta_R(v_{n-1})) \triangleleft BP_*BP$,

$$\begin{aligned} p^{n-1}\eta_R(v_n) \equiv & \\ & \sum_{0\leqslant i\leqslant n} p^{n-i} cp_i t_{n-i}^{p^i} - \sum_{0\leqslant i\leqslant n-1} p^{n-1-i} cp_i t_{n-1-i}^{p^i}\eta_R(v_1)^{p^{n-1}} \\ & \qquad\qquad \bmod (\eta_R(v_2),\ldots,\eta_R(v_{n-1})). \end{aligned}$$

We shall see later that similar phenomena in ℓ_*BP give rise to congruences in $\ell_*\ell$.

We shall now derive some formulae in ℓ_*BP. The natural map of ring spectra $BP \longrightarrow \ell$ is determined on homotopy by

$$v_r \longmapsto \begin{cases} u & \text{if } r = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{9.3}$$

Recalling (6.3), we see that in $\operatorname{im}[H_*\ell \longrightarrow H\mathbb{Q}_*\ell]$, the logarithm series for the factor of ℓ is

$$\log^\ell T = \sum_{n\geqslant 0} \lambda_n T^{p^n} = \sum_{n\geqslant 0} \frac{u^{p^{n-1}+\cdots+p+1}}{p^n} T^{p^n}.$$

We can project (9.2) into ℓ_*BP, with η_R being replaced by the ℓ-theory Hurewicz homomorphism $\underline{\ell}\colon BP_* \longrightarrow \ell_*BP$. This yields

$$\begin{aligned} \underline{\ell}(v_n) &= pt_n - t_{n-1}\underline{\ell}(v_1)^{p^{n-1}} \\ &\quad + \sum_{1\leqslant i\leqslant n} \frac{u^{p^{i-1}+\cdots+p+1} t_{n-i}^{p^i}}{p^{i-1}} - \sum_{1\leqslant i\leqslant n-1} \frac{u^{p^{i-1}+\cdots+p+1} t_{n-1-i}^{p^i}\underline{\ell}(v_1)^{p^{n-1}}}{p^i} \\ &\qquad - \sum_{1\leqslant i\leqslant n-2} t_i\underline{\ell}(v_{n-i})^{p^i} - \sum_{\substack{1\leqslant i\leqslant n-2\\ 1\leqslant j\leqslant i}} \frac{u^{p^{j-1}+\cdots+p+1} t_{i-j}^{p^j}\underline{\ell}(v_{n-i})^{p^i}}{p^j}. \end{aligned}$$

and the equivalent formula

$$\begin{aligned} \underline{\ell}(v_n) &= pt_n + (ut_{n-1}^p - \underline{\ell}(v_1)^{p^{n-1}} t_{n-1}) \\ &\quad + \sum_{1\leqslant i\leqslant n-1} \frac{u^{p^{i-1}+\cdots+p+1}(u^{p^i} t_{n-1-i}^{p^{i+1}} - \underline{\ell}(v_1)^{p^{n-1}} t_{n-1-i}^{p^i})}{p^i} \\ &\quad - \sum_{1\leqslant i\leqslant n-2} t_i\underline{\ell}(v_{n-i})^{p^i} - \sum_{\substack{1\leqslant i\leqslant n-2\\ 1\leqslant j\leqslant i}} \frac{u^{p^{j-1}+\cdots+p+1} t_{i-j}^{p^j}\underline{\ell}(v_{n-i})^{p^i}}{p^j}. \end{aligned} \tag{9.4}$$

Thus we have

$$\begin{aligned}\underline{\ell}(v_2) &= pt_2 + (ut_1^p - \underline{\ell}(v_1)^p t_1) + \frac{u(u^p - \underline{\ell}(v_1)^p)}{p} \\ &= pt_2 + (1-p^{p-1})ut_1^p - \underline{\ell}(v_1)^p t_1 - \sum_{1\leqslant i\leqslant p-1} \binom{p}{i} p^{i-1} u^{p+1-i} t_1^i.\end{aligned}$$

By the Hattori-Stong theorem, the element $\underline{\ell}(v_n) \in \ell_* BP$ is not divisible by p, but notice that on multiplying by p^{n-2} we have

$$\begin{aligned}p^{n-2}\underline{\ell}(v_n) &= p^{n-1}t_n + p^{n-2}(ut_{n-1}^p - \underline{\ell}(v_1)^{p^{n-1}} t_{n-1}) \\ &\quad + \sum_{1\leqslant i\leqslant n-1} \frac{u^{p^{i-1}+\cdots+p+1}(u^{p^i} t_{n-1-i}^{p^{i+1}} - \underline{\ell}(v_1)^{p^{n-1}} t_{n-1-i}^{p^i})}{p^{i-n+2}} \\ &\quad - \sum_{1\leqslant i\leqslant n-2} p^{n-2} t_i \underline{\ell}(v_{n-i})^{p^i} - \sum_{\substack{1\leqslant i\leqslant n-2\\ 1\leqslant j\leqslant i}} p^{n-2-j} u^{p^{j-1}+\cdots+p+1} t_{i-j}^{p^j} \underline{\ell}(v_{n-i})^{p^i}.\end{aligned}$$

and so

$$\begin{aligned}&p^{n-1}t_n + p^{n-2}(ut_{n-1}^p - \underline{\ell}(v_1)^{p^{n-1}} t_{n-1}) \\ &\quad + \sum_{1\leqslant i\leqslant n-1} \frac{u^{p^{i-1}+\cdots+p+1}(u^{p^i} t_{n-1-i}^{p^{i+1}} - \underline{\ell}(v_1)^{p^{n-1}} t_{n-1-i}^{p^i})}{p^{i-n+2}} \\ &\qquad \equiv 0 \bmod (\underline{\ell}(v_2), \ldots, \underline{\ell}(v_n)).\end{aligned}$$

Using the identity $\underline{\ell}(v_1) = u + pt_1$ and the resulting congruences (see Lemma 9.1),

$$\underline{\ell}(v_1)^{p^m} \equiv u^{p^m} \bmod (p^{m+1}) \qquad (m \geqslant 1),$$

we deduce that when $n \geqslant 2$,

$$\begin{aligned}\underline{\ell}(v_n) &\equiv (pt_n - p^{p^{n-1}} t_1^{p^{n-1}} t_{n-1}) + (ut_{n-1}^p - u^{p^{n-1}} t_{n-1}) \\ &\quad + \sum_{1\leqslant i\leqslant n-2} \frac{u^{p^{i-1}+\cdots+p+1}(u^{p^i} t_{n-1-i}^{p^{i+1}} - u^{p^{n-1}} t_{n-1-i}^{p^i})}{p^i} \\ &\quad - \sum_{1\leqslant i\leqslant n-2} t_i \underline{\ell}(v_{n-i})^{p^i} - \sum_{\substack{1\leqslant i\leqslant n-2\\ 1\leqslant j\leqslant i}} \frac{u^{p^{j-1}+\cdots+p+1} t_{i-j}^{p^j} \underline{\ell}(v_{n-i})^{p^i}}{p^j} \quad \bmod (pu).\end{aligned} \tag{9.5}$$

Thus when $n=2$ we have

$$\begin{aligned}\underline{\ell}(v_2) &\equiv (pt_2 - p^p t_1^p t_1) + (ut_1^p - u^p t_1) \quad \bmod (pu)\\ &\equiv ut_1^p - u^p t_1 \quad \bmod (p).\end{aligned}$$

When working in the image of the rationalization map $H_*(\ell \wedge \ell) \longrightarrow H\mathbb{Q}_*(\ell \wedge \ell)$, we shall denote by u and v the images of $u \in \ell_{2p-2}$ under the left and right units for $\ell \wedge \ell$.

Reinterpreting (9.2) in $H\mathbb{Q}_*(\ell \wedge \ell)$, for each $n \geqslant 2$ we have $\eta_R(v_n) \longmapsto 0$ and so

$$\begin{aligned} pt_n + ut_{n-1}^p + \sum_{1\leqslant h\leqslant n-1} \frac{u^{(p^h+p^{h-1}+\cdots+p+1)} t_{n-h-1}^{p^{h+1}}}{p^h} \\ = t_{n-1} v^{p^{n-1}} + \sum_{1\leqslant k\leqslant n-1} \frac{u^{(p^{k-1}+p^{k-2}+\cdots+p+1)} t_{n-1-k}^{p^k} v^{p^{n-1}}}{p^k}. \end{aligned}$$

On rearranging this, we obtain

$$\begin{aligned} pt_n = v^{p^{n-1}} t_{n-1} - ut_{n-1}^p \\ + \sum_{1\leqslant k\leqslant n-1} \frac{u^{(p^{k-1}+p^{k-2}+\cdots+p+1)} (v^{p^{n-1}} t_{n-1-k}^{p^k} - u^{p^k} t_{n-k-1}^{p^{k+1}})}{p^k}. \end{aligned} \tag{9.6}$$

For small values of n we have

$$\begin{aligned} pt_1 &= v - u,\\ pt_2 &= v^p t_1 - ut_1^p + \frac{u(v^p - u^p)}{p},\\ pt_3 &= v^{p^2} t_2 - ut_2^p + \frac{u(v^{p^2} t_1^p - u^p t_1^{p^2})}{p} + \frac{u^{p+1}(v^{p^2} - u^{p^2})}{p^2},\\ pt_4 &= v^{p^3} t_3 - ut_3^p + \frac{u(v^{p^3} t_2^p - u^p t_2^{p^2})}{p} + \frac{u^{p+1}(v^{p^3} t_1^{p^2} - u^{p^2} t_1^{p^3})}{p^2}\\ &\qquad + \frac{u^{p^2+p+1}(v^{p^3} - u^{p^3})}{p^3}. \end{aligned}$$

We want to draw some general conclusions about these expressions.

Lemma 9.3. In $\ell_*\ell$, for $n \geqslant 1$, we have the congruences

$$pt_n \equiv v^{p^{n-1}} t_{n-1} - ut_{n-1}^p \bmod (pu), \tag{9.7}$$

$$pt_n - p^{p^{n-1}} t_1^{p^{n-1}} \equiv u^{p^{n-1}} t_{n-1} - ut_{n-1}^p \bmod (pu). \tag{9.8}$$

Proof. We shall prove this by induction on n, the case $n = 1$ being noted above. So suppose that

$$pt_k \equiv v^{p^{k-1}} t_{k-1} - ut_{k-1}^p \bmod (pu).$$

whenever $1 \leqslant k < n$ for some n. Then for every such k we have

$$v^{p^{k-1}} t_{k-1} \equiv ut_{k-1}^p \bmod (p).$$

By Lemma 9.1, for every $m \geqslant 1$,

$$(v^{p^{k-1}} t_{k-1})^{p^m} \equiv (ut_{k-1}^p)^{p^m} \bmod (p^{m+1}),$$

i.e.,

$$v^{p^{m+k-1}} t_{k-1}^{p^m} \equiv u^{p^m} t_{k-1}^{p^{m+1}} \bmod (p^{m+1}).$$

Now when $1 \leqslant k \leqslant n-1$,

$$v^{p^{n-1}} t_{n-1-k}^{p^k} - u^{p^k} t_{n-k-1}^{p^{k+1}} \equiv 0 \bmod (p^{k+1}),$$

hence in the formula for pt_n in (9.6), the summand

$$u^{(p^{k-1}+p^{k-2}+\cdots+p+1)} \frac{(v^{p^{n-1}} t_{n-1-k}^{p^k} - u^{p^k} t_{n-k-1}^{p^{k+1}})}{p^k}$$

must be divisible by pu. Therefore we have the congruence

$$pt_n \equiv v^{p^{n-1}} t_{n-1} - ut_{n-1}^p \bmod (pu).$$

Using the expansion

$$v^{p^{n-1}} = u^{p^{n-1}} + \sum_{1 \leqslant j \leqslant p^{n-1}} \binom{p^{n-1}}{j} u^{p^{n-1}-j} p^j t_1^j$$

we obtain

$$pt_n - p^{p^{n-1}} t_1^{p^{n-1}} \equiv u^{p^{n-1}} t_{n-1} - ut_{n-1}^p \bmod (pu).$$

Q.E.D.

10 Summary

Kane [6, (19:6:1)], using Adams' criterion [1, III, 17.6], worked out what the image of the torsion-free part of $\ell_*\ell$ is on passage to $\mathbb{Q} \otimes \ell_*\ell$. As a $\pi_*\ell$-module the torsion-free part of $\ell_*\ell$ is generated by the elements

$$t_{n,i} = \frac{u^i v(v-(p-1)u)\cdots(v-(n-1)(p-1)u)}{p^i} \qquad (0 \leqslant i \leqslant \nu_p(n!)).$$

Obviously, the relation $ut_{n,i} = pt_{n,i+1}$ holds and it is clear how to multiply elements of that form.

To summarize our results on the multiplicative structure of $\ell_*\ell$, we have the following:

- Starting with two non-torsion elements in $\ell_*\ell$, we can consider their images in $\mathbb{Q} \otimes \ell_*\ell$, take their product there and interpret the result as a non-torsion element in $\ell_*\ell$.
- Any two elements coming from the zero-line of the Künneth spectral sequence multiply according to the congruence relations we specified Lemma 9.3. These elements might be torsion or non-torsion, but there is no non-torsion in higher filtrations.
- Torsion elements in non-zero filtration have their origin in the generators Δ_u and for these we spelled out the multiplication in (5.2a).
- As the Δ_u-expressions allow coefficients from ℓ_*BP, the multiplication of non-torsion elements in the zero-line with torsion elements in higher filtration is determined as well.

We agree that the recursive nature of the congruences for $\ell_*BP \otimes_{BP_*} \ell_*$ might hamper the calculation, but our approach leads to more information about the multiplication in $\ell_*\ell$ than the known sources.

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Appendix

A Regular sequences in Hopf algebroids

In this appendix we give some results on regular sequences in Hopf algebroids that we make use of in several places. Although these may be well known, we feel that it is convenient to highlight them as we do not know any convenient appearance of them in the literature.

Let R be a commutative ring and let (A,Γ) be a Hopf algebroid over R (see for instance [15, Appendix A1]). We can view Γ as a commutative A-algebra using the left unit $A \longrightarrow \Gamma$, and the right unit provides a second ring homomorphism $\eta\colon A \longrightarrow \Gamma$ which we use to give Γ a right A-module structure; together these make Γ into an A-bimodule. These two homomorphisms $A \longrightarrow \Gamma$ are interchanged by the antipode $\chi\colon \Gamma \longrightarrow \Gamma$ and equalised by the counit $\varepsilon\colon \Gamma \longrightarrow A$. If Γ is flat as a left (or equivalently right) A-module, we shall say that the Hopf algebroid (A,Γ) is *flat*.

Given a left A-module M, we can define the A-bimodule tensor product $\Gamma \otimes_A M$ which has a natural left Γ-comodule structure with coproduct

$$\psi\colon \Gamma \otimes_A M \xrightarrow{\psi_\Gamma \otimes \mathrm{id}} \Gamma \otimes_A \Gamma \otimes_A M.$$

This construction is natural in the A-module M.

Recall from [11] that an element $a \in A$ is said to be *M-regular* if the multiplication by a map on M has trivial kernel. More generally, a (possibly infinite) sequence $a_1, a_2, \ldots$ in A is *M-regular* if a_1 is M-regular and for each $n > 1$ where a_n exists, a_n is $M/(a_1,\ldots,a_{n-1})M$-regular. When $M = A$, we say that such a sequence is *regular*.

Let $A \longrightarrow B$ be a homomorphism of commutative R-algebras. We say that B is *Landweber exact* with respect to (A,Γ) if $B \otimes_A (-)$ is an exact functor on Γ-comodules.

Theorem A.1. Assume that (A,Γ) is a flat Hopf algebroid and that M is a left A-module.
(i) Let $a \in A$. Then multiplication by a on M induces a morphism of Γ-comodules

$$\Gamma \otimes_A M \xrightarrow{\mathrm{id}\otimes a} \Gamma \otimes_A M.$$

If a is M-regular, then there is a short exact sequence of Γ-comodules

$$0 \to \Gamma \otimes_A M \xrightarrow{\mathrm{id}\otimes a} \Gamma \otimes_A M \to \Gamma \otimes_A M/aM \to 0.$$

In particular, when $M = A$ there is an isomorphism of Γ-comodules

$$\Gamma/(\eta(a)) \cong \Gamma \otimes_A A/(a).$$

(ii) If $a_1, a_2, \ldots$ is a regular sequence in A, then each of $a_1, a_2, \ldots$ and $\eta(a_1), \eta(a_2), \ldots$ is a regular sequence in Γ. For each n for which a_n exists, there is a short exact sequence of Γ-comodules

$$0 \to \Gamma/(\eta(a_1), \ldots \eta(a_{n-1})) \xrightarrow{\eta(a_n)} \Gamma/(\eta(a_1), \ldots \eta(a_{n-1})) \\ \to \Gamma/(\eta(a_1), \ldots, \eta(a_n)) \to 0.$$

(iii) Let $A \longrightarrow B$ be a homomorphism of commutative R-algebras for which $B \otimes_A (-)$ is exact on Γ-comodules. If $a_1, a_2, \ldots$ is a regular sequence in A, then $1 \otimes \eta(a_1), \ldots, 1 \otimes \eta(a_n)$ is a regular sequence in $B \otimes_A \Gamma$.

Proof. (i) The first statement is clear since the tensor product satisfies

$$\gamma \otimes am = \gamma\eta(a) \otimes m \quad (\gamma \in \Gamma, m \in M),$$

and since $\eta(a)$ is primitive,

$$\psi(\gamma\eta(a)) = \psi(\gamma)(1 \otimes \eta(a)).$$

When a acts regularly, on applying the functor $\Gamma \otimes_A (-)$ to the short exact sequence of A-modules

$$0 \to M \xrightarrow{a} M \longrightarrow M/aM \to 0$$

and using the flatness of Γ as a right A-module, we obtain the desired exact sequence. Parts (ii) and (iii) follow from (i). Q.E.D.

Corollary A.2. Suppose that $a_0, a_1, \ldots$ is a regular sequence in A and that $\eta(a_0) = a_0$. Then $(A/(a), \Gamma/(a))$ is a flat Hopf algebroid and the sequence $\bar{a}_1, \bar{a}_2, \ldots$ in $A/(a)$ is regular, hence the sequences $a_0, a_1, \ldots$ and $\eta(\bar{a}_1), \eta(\bar{a}_2), \ldots$ are regular in $\Gamma/(a) = \Gamma/(\eta(\bar{a}))$.

Proof. The flatness is easily verified and the rest follows from (ii). Q.E.D.

Remark A.3. As a particular example of the phenomenon described in (iii), we have the case of a Landweber exact complex oriented commutative ring spectrum E whose the homology theory satisfies

$$E_*(-) \cong E_* \otimes_{MU_*} MU_*(-).$$

Then for a regular sequence $x_1, x_2, \ldots$ in MU_*, the sequence of Hurewicz images $\underline{e}(x_1), \underline{e}(x_2), \ldots$ is regular in E_*MU. For example we might take each $x_n \in MU_{2n}$ to be a polynomial generator.

B Toda brackets in Künneth spectral sequences

As the details of the argument we require for convergence of certain triple Massey products to Toda brackets in the Künneth spectral sequence are not in the literature, we follow the referee's suggestion and give an account of what is needed, making allowances for differences in gradings, etc. This is based on [7, Proposition 5.4.5] and its following paragraph on homology spectral sequences, as well as [7, § 5.7]. This material fits into the more general framework of [12, Theorem 4.1]. For completeness we begin by discussing Massey products and Toda brackets in Künneth spectral sequences of the type we are using, thus refining a result on multiplicative structure of [3].

The following observation is well known: If B is a commutative ring and if C is a B-algebra, then there is a quasi-isomorphism of non-negatively graded differential graded B-algebras (B-dgas) $P_\bullet \longrightarrow C$, where C is regarded as a differential graded algebra concentrated in degree 0 and each P_s is a free B-module.

Suppose that R is a connective commutative $\mathbb{S}$-algebra and that A, B are two connective R-algebras. If B is q-cofibrant, then the Künneth spectral sequence converging to $A^R_*B = \pi_*(A \wedge_R B)$ is constructed by taking a free resolution $P_{\bullet,*} \longrightarrow A_*$ over R_* and realising each $P_{s,*}$ as π_*P_s, where P_s is a wedge of sphere R-modules, with the boundaries induced from maps of R-modules $P_s \longrightarrow P_{s-1}$. It was pointed out in [3, Lemma 1.3] that it was always possible to produce a product structure on $P_\bullet$ with product maps $P_s \wedge_R P_t \longrightarrow P_{s+t}$. However, we need to do this in a more precise way by ensuring that $P_{\bullet,*}$ is actually a R_*-dga.

Proposition B.1. For a connective commutative $\mathbb{S}$-algebra R and two connective R-algebras A, B where B is q-cofibrant as an R-module, the Künneth spectral sequence $(\mathrm{E}^r_{s,t}, d^r)$ is a spectral sequence of R_*-dgas

$$\mathrm{E}^2_{s,t} = \mathrm{Tor}^{R_*}_{s,t}(A_*, B_*) \Longrightarrow A^R_{s+t}B = \pi_{s+t}(A \wedge_R B), \qquad \text{(KSS)}$$

with differentials $d^r\colon \mathrm{E}^r_{s,t} \longrightarrow \mathrm{E}^r_{s-r,t+r-1}$.

Proof. Since we need details in considering Massey products and Toda brackets in the spectral sequence, we recall its construction, in particular emphasising the multiplicative aspects.

Take an R_*-dga resolution $P_{\bullet,*}$ of A_* as above and realise each $P_{s,*}$ as the homotopy of a wedge of R-spheres P_s, so $P_{s,*} = \pi_*P_s$. From the construction in the proof of [3, Lemma 1.3], there is a directed system of cofibrations of R-modules

$$\begin{tikzcd} A'_0 \arrow[r, "i'_1"] \arrow[drrr, "i^0"'] & A'_1 \arrow[r, "i'_2"] \arrow[drr, "i^1"] & \cdots \arrow[r, "i'_s"] & A'_s \arrow[r, "i'_{s+1}"] \arrow[d, "i^s"] & \cdots \\ & & & A & \end{tikzcd}$$

where $\operatorname{hocolim}_s A'_s$ is equivalent to A, and for which there are associated cofibre sequences

$$A'_{s-1} \xrightarrow{i'_s} A'_s \xrightarrow{q'_s} \Sigma^s P_s.$$

These are multiplicative in the sense that there are maps $\mu'_{s_1,s_2} \colon A'_{s_1} \wedge_R A'_{s_2} \longrightarrow A'_{s_1+s_2}$ and commutative diagrams in the homotopy category

$$\begin{tikzcd} A'_{s_1} \wedge_R A'_{s_2} \arrow[r, "\mu'_{s_1,s_2}"] \arrow[d, "i^{s_1} \wedge i^{s_2}"'] & A'_{s_1+s_2} \arrow[d, "i^{s_1+s_2}"] \\ A \wedge_R A \arrow[r] & A \end{tikzcd}$$

in which unlabelled maps are the evident ones. Writing $W_s = A'_s \wedge_R B$, we obtain further cofibre sequences

$$W_{s-1} \xrightarrow{i_s} W_s \xrightarrow{q_s} \Sigma^s P_s \wedge_R B,$$

and on applying homotopy we obtain long exact sequences with boundary maps

$$\partial_s \colon \pi_*(\Sigma^s P_s \wedge_R B) \longrightarrow \pi_{*-1} W_{s-1}.$$

We also have that $W_\infty = \operatorname{hocolim}_s W_s$ is equivalent to $A \wedge_R B$.

The spectral sequence is set up by setting

$$\mathrm{E}^1_{s,t} = \pi_{s+t}(\Sigma^s P_s \wedge_R B) \cong \pi_t(P_s \wedge_R B)$$

and taking d^1 to be the composition

$$d^1 \colon \mathrm{E}^1_{s,t} = \pi_{s+t}(\Sigma^s P_s \wedge_R B) \xrightarrow{\partial_s} \pi_{s+t-1} W_{s-1} \xrightarrow{(q_{s-1})_*} \pi_{s+t-1}(\Sigma^{s-1} P_{s-1} \wedge_R B) = \mathrm{E}^1_{s-1,t}.$$

As the maps $i_s \colon W_{s-1} \longrightarrow W_s$ are cofibrations, $\mathrm{E}^1_{s,t}$ can be identified with a relative homotopy group,

$$\mathrm{E}^1_{s,t} = \pi_{s+t}(W_s, W_{s-1}),$$

where for a cofibration $Y \longrightarrow X$, $\pi_n(X,Y)$ denotes the homotopy classes of maps of pairs $(D^n, S^{n-1}) \longrightarrow (X,Y)$; here we abuse notation by writing (D^n, S^{n-1}) for the pair of R-modules $(\mathbb{F}_R D^n, \mathbb{F}_R S^{n-1})$ consisting of the free R-modules on the disc and sphere spectra, respectively. Guided by the discussion in [7, § 5.7], in the following we shall make systematic use of this interpretation.

There is a product structure on the directed system of W_s's, giving homotopy commutative diagrams

$$\begin{tikzcd} W_{s_1} \wedge_R W_{s_2} \arrow[r, "\mu_{s_1,s_2}"] \arrow[d] & W_{s_1+s_2} \arrow[d] \\ W_\infty \wedge_R W_\infty \arrow[r] & W_\infty \end{tikzcd}$$

giving rise to a product in the spectral sequence compatible with the dga structure on the resolution $P_{\bullet,*}$. Q.E.D.

We recall some facts about the spectral sequence (KSS), all of which can be deduced by analogy with the case considered in [7]. This spectral sequence is homologically graded and its target has an increasing filtration

$$0 \subseteq F_0 A_n^R B \subseteq F_1 A_n^R B \subseteq \cdots \subseteq F_n A_n^R B = A_n^R B$$

for which

$$F_s A_n^R B / F_{s-1} A_n^R B \cong \mathrm{E}^\infty_{s,n-s}.$$

For $r \geqslant 1$,

$$\mathrm{E}^r_{s,t} = \frac{\operatorname{im} i_* \colon \pi_{s+t} W_s / W_{s-r} \longrightarrow \pi_{s+t} W_s / W_{s-1}}{\operatorname{im} \partial \colon \pi_{s+t+1} W_{s+r-1} / W_s \longrightarrow \pi_{s+t} W_s / W_{s-1}}, \tag{B.1}$$

where the maps are the evident ones obtained by composing maps between W_k's and associated boundaries. Similarly,

$$\mathrm{E}^\infty_{s,t} = \frac{\operatorname{im} i_* \colon \pi_{s+t} W_s \longrightarrow \pi_{s+t} W_s / W_{s-1}}{\operatorname{im} \partial \colon \pi_{s+t+1} W_\infty / W_s \longrightarrow \pi_{s+t} W_s / W_{s-1}}. \tag{B.2}$$

We recall from Kochman's definition 5.4.1 what it means for a Massey product $\langle x_1, x_2, x_3 \rangle$ to be *defined* in $\mathrm{E}^{r+1}_{*,*} = H(\mathrm{E}^r_{*,*}, d^{r-1})$, where x_1, x_2, x_3 are elements of $\mathrm{E}^r_{*,*}$. The following conditions must hold: there is a *defining system* for $\langle x_1, x_2, x_3 \rangle$

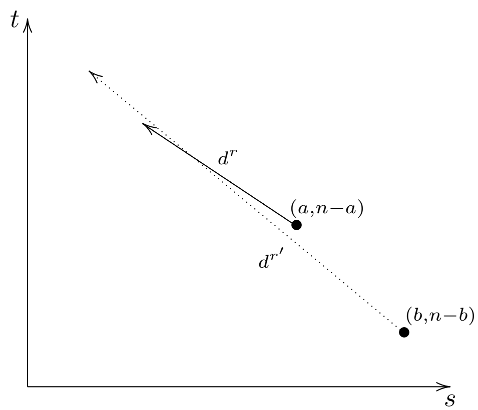
$$\begin{array}{ccccc} X_{0,1} & & X_{1,2} & & X_{2,3} \\ & X_{0,2} & & X_{1,3} & \end{array}$$

consisting of elements $X_{i,j}$ of $\mathrm{E}^r_{*,*}$, where $X_{0,1}, X_{1,2}, X_{2,3}$ are cycles representing $x_1 = [X_{0,1}]$, $x_2 = [X_{1,2}]$, $x_3 = [X_{2,3}]$ and

$$d^r X_{0,2} = \bar{X}_{0,1} X_{1,2}, \quad d^r X_{1,3} = \bar{X}_{1,2} X_{2,3}.$$

Here $\bar{Z} = (-1)^{s+t+1} Z$ if $Z \in \mathrm{E}^r_{s,t}$. Then $\langle x_1, x_2, x_3 \rangle \subseteq \mathrm{E}^{r+1}_{*,*}$ is the subset consisting of all homology classes $[\bar{X}_{0,1} X_{1,3} + \bar{X}_{1,2} X_{2,3}]$ obtained from all possible defining systems of $\langle x_1, x_2, x_3 \rangle$.

Given $u \in \mathrm{E}^r_{a,n-a}$ and the relation $d^r u = x$, then a differential of the form $d^{r'} w = y$ with $w \in \mathrm{E}^{r'}_{b,n-b}$ and $a < b$ is said to be a *crossing differential* of $d^r u = x$ if $a + r > b + r'$.



Guided by Kochman's account [7, § 5.7], we recall the definition of a Toda bracket of the form $\langle \alpha, \beta, \gamma \rangle$ in the homotopy of an R ring spectrum E. We can make use of the monoidal smash product on the category of R-modules to simplify some of the details. Suppose that

$$\alpha = [g_{0,1}] \in \pi_a E, \quad \beta = [g_{1,2}] \in \pi_b E, \quad \gamma = [g_{2,3}] \in \pi_c E,$$

and

$$\alpha\beta = 0 = \beta\gamma.$$

If we choose null-homotopies $g_{0,2} \colon D^{a+b+1} \longrightarrow E$ and $g_{1,3} \colon D^{b+c+1} \longrightarrow E$ for $g_{0,1} g_{1,2}$ and $g_{1,2} g_{2,3}$, then

$$\begin{array}{ccccc} g_{0,1} & & g_{1,2} & & g_{2,3} \\ & g_{0,2} & & g_{1,3} & \end{array}$$

is a defining system for $\langle \alpha, \beta, \gamma \rangle \subseteq \pi_{a+b+c+1}$ and using Kochman's notation, we denote the homotopy classes that constitute the Toda bracket by

$$[\bar{g}_{0,2} g_{2,3} \cup \bar{g}_{0,1} g_{1,3}]$$

obtained from all choices of null-homotopies, where we glue two copies of $D^{a+b+c+1}$ along their boundaries to form a sphere, and also use the above sign convention to determine $\bar{g}$ as $\pm g$.

Theorem B.2. Assume that the following conditions hold in the spectral sequence (KSS).

- The elements x_1, x_2, x_3 in $\mathrm{E}^r_{*,*}$ are infinite cycles which converge to the elements ξ_1, ξ_2, ξ_3 in $A^R_* B$.
- The Massey product $\langle x_1, x_2, x_3\rangle$ is defined in $\mathrm{E}^{r+1}_{*,*}$.
- The Toda bracket $\langle \xi_1, \xi_2, \xi_3\rangle$ is defined in $A^R_* B$.
- If $X_{i,j}$ is a defining system for $\langle x_1, x_2, x_3\rangle$, then there are no crossing differentials for the differentials $d^r X_{0,2} = \bar{X}_{0,1} X_{1,2}$ and $d^r X_{1,3} = \bar{X}_{1,2} X_{2,3}$.

Then $\langle x_1, x_2, x_3\rangle$ is a set of infinite cycles which converge to elements of $\langle \xi_1, \xi_2, \xi_3\rangle$ in $A^R_* B$.

Proof. We adapt the ideas in the proof of [7, Proposition 5.7.5] in the simplest case of a triple Toda bracket, making necessary changes to accommodate differences in gradings and signs. We use the notation established above, in particular we write $W_\infty = \operatorname{hocolim}_s W_s \sim A \wedge_R B$. Of course, the argument can be extended to work for Toda brackets of arbitrary length.

For each pair (i, j) and fixed r, let

$$\gamma_{i,j} \colon (W_{s(i,j)}, W_{s(i,j)-r}) \longrightarrow (W_{s(i,j)}, W_{s(i,j)-1})$$

be the obvious map of pairs.

Let $X_{i,j} \in \mathrm{E}^r_{s(i,j),t(i,j)}$ be a defining system for $\langle x_1, x_2, x_3\rangle$. We shall produce a defining system

$$\begin{array}{ccccc} \xi_{0,1} & & \xi_{1,2} & & \xi_{2,3} \\ & \xi_{0,2} & & \xi_{1,3} & \end{array}$$

for $\langle \xi_1, \xi_2, \xi_3\rangle$, so that each $\xi_{i,j}$ represents $X_{i,j}$ in the spectral sequence.

Since ξ_i represents $x_i \in \mathrm{E}^r_{s(i-1,i),t(i-1,i)}$, we may choose an element

$$\Xi_{i-1,i} \colon (D^{s(i-1,i)+t(i-1,i)}, S^{s(i-1,i)+t(i-1,i)-1}) \longrightarrow (W_{s(i-1,i)}, W_{s(i-1,i)-r})$$

for which

$$\gamma_{i-1,i}\Xi_{i-1,i} \colon (D^{s(i-1,i)+t(i-1,i)}, S^{s(i-1,i)+t(i-1,i)-1}) \\ \longrightarrow (W_{s(i-1,i)}, W_{s(i-1,i)-1})$$

represents $X_{i-1,i} \in \mathrm{E}^r_{s(i,j),t(i,j)}$, where we make use of the isomorphism of (B.1). Now choose maps

$$\Xi'_{0,2}\colon (D^{s(0,2)+t(0,2)}, S^{s(0,2)+t(0,2)-1}) \longrightarrow (W_{s(0,2)}, W_{s(0,2)-r}),$$
$$\Xi'_{1,3}\colon (D^{s(1,3)+t(1,3)}, S^{s(1,3)+t(1,3)-1}) \longrightarrow (W_{s(1,3)}, W_{s(1,3)-r})$$

to represent $X_{0,2}, X_{1,3}$, respectively. By assumption on the $X_{0,2}, X_{1,3}$, each of the precompositions

$$(\bar{\Xi}_{i-2,i-1}\Xi_{i-1,i} \vee -\Xi'_{i-2,i})|_{S^{s(i-2,i)+t(i-2,i)-1} \vee S^{s(i-2,i)+t(i-2,i)-1}}$$

with the pinch map

$$\nabla\colon S^{s(i-2,i)+t(i-2,i)-1} \longrightarrow S^{s(i-2,i)+t(i-2,i)-1} \vee S^{s(i-2,i)+t(i-2,i)-1}$$

factors (up to homotopy) through a map

$$\lambda'_i\colon S^{s(i-2,i)+t(i-2,i)-1} \longrightarrow W_{s(i-2,i)-r-1}.$$

Since the increasing filtration on $\pi_n W$ satisfies $F_{-1}\pi_n W = 0$, there must be two maps

$$\alpha_i\colon (D^{s(i-2,i)+t(i-2,i)}, S^{s(i-2,i)+t(i-2,i)-1}) \longrightarrow (W_{s(i-2,i)-1}, W_{s(i-2,i)-r-1})$$

such that in each case one of the following possibilities has to occur:

- $(\alpha_i)|_{S^{s(i-2,i)+t(i-2,i)-1}} = \lambda'_i$,
- $(\lambda'_i \vee -\alpha_i)|_{S^{s(i-2,i)+t(i-2,i)-1} \vee S^{s(i-2,i)+t(i-2,i)-1}}$ precomposed with the pinch map ∇ factors through a map

  $$\lambda_i\colon S^{s(i-2,i)+t(i-2,i)-1} \longrightarrow W_{s(i-2,i)-r-m}$$

  with $m \geqslant 2$ as large as possible.

If the latter case occurred, λ_i would bound since $\xi_1\xi_2 = 0 = \xi_2\xi_3$, implying the existence of a map

$$\beta_i\colon (D^{s(i-2,i)+t(i-2,i)}, S^{s(i-2,i)+t(i-2,i)-1}) \longrightarrow (W_{s(i-2,i)-k}, W_{s(i-2,i)-r-m})$$

with k minimal. But this defines a non-trivial d^{r+m-k} boundary, which is a crossing differential for $d^r X_{i-2,i} = \bar{X}_{i-2,i-1}X_{i-1,i}$. Since no such crossing differentials can exist, the first possibility must occur in each case and we set $\xi_{i-2,i} = \Xi'_{i-2,i} \cup -\alpha_i$.

Thus we can construct a defining system $\xi_{i,j}$ for $\langle \xi_1, \xi_2, \xi_3 \rangle$ which lifts the defining system $X_{i,j}$ for $\langle x_1, x_2, x_3 \rangle$. Q.E.D.