Theories with Ehrenfeucht-Fraïssé equivalent non-isomorphic models

Saharon Shelah*

Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Edmond J. Safra Campus, Givat Ram, Jerusalem 91904, Israel

Department of Mathematics, The State University of New Jersey, Rutgers, Hill Center-Busch Campus, 110 Frelinghuysen Road, Piscataway NJ, 08854-8019, United States of America

E-mail: `shelah@math.huji.ac.il`

Abstract

Our "long term and large scale" aim is to characterize the first order theories T (at least the countable ones) such that for every ordinal α there are λ, M_1, M_2 such that M_1 and M_2 are non-isomorphic models of T of cardinality λ which are $\mathrm{EF}^+_{\alpha,\lambda}$-equivalent. We expect that as in the main gap [11, XII], we get a strong dichotomy, i.e., on the non-structure side we have stronger, better examples, and on the structure side we have an analogue of [11, XIII]. We presently prove the *consistency* of the non-structure side for T which is $\aleph_0$-independent (= not strongly dependent), even for $\mathrm{PC}(T_1, T)$.

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1 Introduction

1.1 Motivation

We first give some an introduction for non-model theorists. A major theme in the author's work in model theory is to find "main gap theorems". This means finding a dichotomy for the family of elementary classes (e.g., the classes of the form $\mathrm{Mod}_T = \{M\,;\, M \models T\}$ for some (complete) first order theory T) such that each such class is either "very simple" or "very complicated". The motivation for this is that we expect to have much knowledge to gain on the "very simple" ones.

Of course, this depends on the criterion for "very simple". The main theorem of [11] does this essentially for countable T, with "very complicated" interpreted as "the number of models in Mod_T of cardinality λ is

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maximal, i.e., 2^λ, for every λ".[1] Here we are interested with interpreting "very complicated" as "for arbitrarily large cardinals, there are models $M_1, M_2 \in \mathrm{Mod}_T$ of cardinality λ which are *very similar* but not isomorphic", where "very similar" is interpreted as equivalent in the sense of Ehrenfeucht-Fraïssé games: These games have two players, the isomorphism player and the anti-isomorphism player. The isomorphism player constructs during the play, partial isomorphism of cardinality $< \lambda$, in each move the anti-isomorphism player demands some elements to be in the domain or the range, the isomorphism player has to extend the partial isomorphism accordingly; in the play there are α moves, $\alpha < \lambda$; and the isomorphism player wins the play if he has a legal move in each stage (cf. Definitions 2.3 and 2.5).

In the present paper we aim at finding the right variant of Ehrenfeucht-Fraïssé game that allows us to interpret "very complicated" as intended (cf. the discussion after Definition 2.3); we then give quite weak sufficient conditions for Mod_T being complicated. Let $T \subseteq T_1$ be complete first order theories. We denote by $\mathrm{PC}(T_1, T)$ the set of reducts of models of T_1 in the language of T. We aim to show: If T is not strongly stable, α is an ordinal and $\lambda > |T|$ (or at least for many such λs) then there are $M_1, M_2 \in \mathrm{PC}(T_1, T)$ of cardinality λ which are $\mathrm{EF}^+_{\alpha,\lambda}$-equivalent for every $\alpha < \lambda$ but not isomorphic.[2]

1.2 Related Work

This paper continues the work of [13] and [1]. For a history of this research area, cf. [19]. Recently, the author gave a new construction in [14] covering also $\aleph_1$; but, whereas it applies to every regular uncountable λ, it seems less amenable to generalizations.

By [11], for a countable complete first order theory T, we essentially know when there are $\mathcal{L}_{\infty,\lambda}(\tau_T)$-equivalent non-isomorphic models of T of cardinality λ for some λ: this is exactly when T is superstable with the NDOP and the NOTOP.[3] Instead of the property "$\mathcal{L}_{\infty,\lambda}(\tau_T)$-equivalent non-isomorphic", we can consider "$\mathrm{EF}_{\alpha,\lambda}$-equivalent non-isomorphic". This investigation was started by Hyttinen and Tuuri [5], and continued by Hyttinen and the present author [2, 3, 4]. In this paper, we shall replace "$\mathrm{EF}_{\alpha,\lambda}$-equivalent non-isomorphic" by a technical variant "$\mathrm{EF}^+_{\alpha,\lambda}$-equivalent non-isomorphic" (cf. Definition 2.5). By [2], if T is a stable, unsuper-

[1] More information on the mentioned theorem can be found, e.g., in [17].

[2] For the definition of $\mathrm{EF}^+_{\alpha,\lambda}$, cf. Definition 2.5 below. This is a somewhat stronger relative of the standard notion of being $\mathrm{EF}_{\alpha,\lambda}$-equivalent.

[3] Cf. § 5 and [9]. In this paper, we shall be referring regularly to the standard notions of classification theory, the "dimensional order property" (DOP) and the "omitting types order property" (OTOP). Theories without these properties are called NDOP and NOTOP, respectively. Definitions will be given in § 1.3.

stable, complete first order theory, $\lambda = \mu^+, \mu = \mathrm{cf}(\mu) \geq |T|$, then there are $\mathrm{EF}_{\mu\times\omega,\lambda}$-equivalent non-isomorphic models of T (even in $\mathrm{PC}(T_1, T)$) of cardinality λ. By our new variant $\mathrm{EF}^+_{\alpha,\lambda}$-equivalent, such results are excluded; by it we define our choice test problem the version of being fat/lean, cf. the definitions in § 1.3.

Among the variants of strongly dependent theories (cf. [6, §3], [7, 18] and [7, §5]), the best relative for us is "strongly$_4$ dependent". We define this below (Definition 3.6), but we delay the treatment to a subsequent paper, [8], where we also deal with the relevant logics and more.

We prove here that if T is not strongly stable then T is consistently fat. More specifically, for every $\mu = \mu^{<\mu} > |T|$ there is a μ-complete class forcing notion $\mathbb{P}$ such that in $\mathbf{V}^{\mathbb{P}}$ the theory T is fat. The result holds even for $\mathrm{PC}(T_1, T)$. This gives new cases even for $\mathrm{PC}(T)$ by Example 1.1.

Also if T is unstable or has the DOP or OTOP (cf. the definitions in § 1.3 or [11]) then it is fat, i.e., already in $\mathbf{V}$.

Of course, it is not optimal to have to force the example, but note that such a result is certainly enough for proving there is no positive theory. Hence it gives us an upper bound on the relevant dividing lines.

1.3 Notation and basic definitions

Let us fix our model-theoretic notation. We fix a first-order theory T. By $\mathrm{Mod}_T(\lambda) = \mathrm{EC}_T(\lambda)$ we denote the class of models of T of cardinality λ; $\mathrm{Mod}_T = \mathrm{EC}_T := \bigcup\{\mathrm{EC}_T(\lambda) : \lambda \text{ a cardinality}\}$. If T is a theory or a sentence in a vocabulary $\tau_T \supseteq \tau$, we write $\mathrm{PC}_\tau(T) = \{M{\restriction}\tau : M \text{ a model of } T\}$ (and if $\tau = \tau_T$ we may omit τ). If $T \subseteq T_1$ are complete first order theories then $\mathrm{PC}(T_1, T) = \mathrm{PC}_{\tau(T)}(T_1)$.

If $\bar{a}$ is a sequence, we denote its length by $\mathrm{lh}(\bar{a})$; by $\bar{a} \trianglelefteq \bar{b}$ we mean that $\bar{a}$ is an initial segment of $\bar{b}$; and by $\bar{a}{\restriction}\alpha$ we denote the unique initial segment of $\bar{a}$ of length α for $\alpha \leq \mathrm{lh}(\bar{a})$.

For regular $\lambda > \aleph_0$, we say that (E, u) *is a witness for* S if (a) E is a club of the regular cardinal λ; (b) $u = \langle u_\alpha : \alpha < \lambda\rangle, a_\alpha \subseteq \alpha$ and $\beta \in a_\alpha \Rightarrow a_\beta = \beta \cap a_\alpha$; and (c) for every $\delta \in E \cap S, u_\delta$ is an unbounded subset of δ of order-type $< \delta$ (and δ is a limit ordinal). For a regular uncountable cardinal λ let $\check{I}[\lambda] = \{S \subseteq \lambda : \text{some pair } (E, \bar{a}) \text{ is a witness for } S\}$.

If I is a linearly ordered set, we let $\mathrm{incr}_\alpha(I) := \{\rho : \rho \text{ is an increasing sequence of length } \alpha \text{ of members of } I\}$; similarly $\mathrm{incr}_{<\alpha}(I) := \bigcup\{\mathrm{incr}_\beta(I) : \beta < \alpha\}$. So instead of $[I]^{<\aleph_0}$ we may use $\mathrm{incr}_{<\omega}(I)$.

For a model $M, \bar{a} \in {}^\alpha M, B \subseteq M$ and Δ a set of formulas, we are interested in formulas of the form $\varphi(\bar{x}, \bar{y})$, $\bar{x} = \langle x_i : i < \alpha\rangle$. Here, α may be infinite, but the formulas are normally first order, so all but finitely many of the x_i's are dummy variables. We write $\mathrm{tp}_\Delta(\bar{a}, B, M) := \{\varphi(\bar{x}, \bar{a}) : \varphi(\bar{x}, \bar{y}) \in \Delta \text{ and } \bar{b} \in {}^{\mathrm{lh}(\bar{y})}A \text{ and } M \models \varphi[\bar{a}, \bar{b}]\}$.

Let $\Delta_{\rm qf}$ be the set of quantifier-free formulas in $\mathcal{L}(\tau_M)$ and write $\mathrm{tp}_{\rm qf}$ instead of $\mathrm{tp}_{\Delta_{\rm qf}}$. By $\dot{I}(\lambda, T)$ we denote the number of isomorphism types of models of T of cardinality λ; by $\dot{I}_\tau(\lambda, T)$ we denote the number of isomorphism types of $M{\restriction}\tau$ for a model $M \models T$ of cardinality λ; and by $\dot{I}\dot{E}_\tau(\lambda, T)$ we denote the supremum of $\{|K| : K \subseteq \mathrm{PC}_\tau(T)$, all $M \in K$ have cardinality λ, and no $M \in K$ has an elementary embedding into any $N \in K \backslash \{M\}\}$; finally, we write $\dot{I}\dot{E}_\tau(\lambda, T) =^+ \chi$ if the supremum is obtained if not said otherwise. We let $\dot{I}\dot{E}(\lambda, T) := \dot{I}\dot{E}_{\tau(T)}(\lambda, T)$.

Let T be a first order complete theory. We say that T has the OTOP ("omitting types order property") when T is stable and for some n, m letting $\bar{x} = \langle x_\ell : \ell < n\rangle, \bar{y} = \langle y_\ell : \ell < n\rangle, \bar{z} = \langle z_\ell : \ell < m\rangle$, there are complete types $p(\bar{x}, \bar{y}, \bar{z})$ such that for every λ there is a model M of T and $\bar{a}_\alpha \in {}^n M$ for $\alpha < \lambda$ such that $\langle \bar{a}_\alpha : \alpha < \lambda\rangle$ is an indiscernible set and for $\alpha \neq \beta < \lambda$ the type $p(\bar{a}_\alpha, \bar{a}_b, \bar{z})$ is realized in M iff $\alpha < \beta$. We say that T has the NOTOP when it is stable but fails to have the OTOP.

We say that T has the DOP ("dimensional order property") when T is stable and we can find $|T|^+$-saturated models M_ℓ of T for $\ell \leq 3$ such that $M_0 \prec M_\ell \prec M_3$ for $\ell = 1, 2$ and $\mathrm{tp}(M_1, M_2)$ does not fork over M_0, M_3 is $|T|^+$-prime over $M_1 \cup M_2$ but not $|T|^+$-minimal over it; equivalently for every $\bar{c} \in {}^{\omega>}(M_3)$ the type $\mathrm{tp}(\bar{c}, M_1 \cup M_2, M_3)$ is $|T|^+$-isolated but there is no infinite $I \subseteq M_3$ which is indiscernible over $M_1 \cup M_2$. We say that T has NDOP when T is stable and fails to have the DOP.

Furthermore, we say T is *fat* when for every ordinal κ, for some (regular) cardinality $\lambda > \kappa$ there are non-isomorphic models M_1, M_2 of T of cardinality λ which are $\mathrm{EF}^+_{\beta,\kappa,\kappa,\lambda}$-equivalent for every $\beta < \lambda$ (cf. Definition 2.5 below). If T is not fat, we say it is *lean*. We say the pair (T, T_1) is fat/lean when ($T_1 \supseteq T$ is a first order theory and) $\mathrm{PC}(T_1, T) := \{M{\restriction}\tau_T : M$ a model of $T_1\}$ is as above. We write that $(T, *)$ *is fat* when for every first order $T_1 \supseteq T$ the pair (T, T_1) is fat. We say that $(T, *)$ *is lean* otherwise.

The results in this paper (mainly Theorem 4.1) seem to cover cases of stable T with the NDOP and the NOTOP. But there are more examples (cf. [7, §5] for details):

Example 1.1.

1. There is a stable countable complete theory with the NDOP and the NOTOP which is not strongly dependent; (moreover not is not strongly$_4$ stable), cf. [7, §5](G).

2. $T = \mathrm{Th}({}^{\omega_1}(\mathbb{Z}_2), E_n)_{n<\omega}$ is as above where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ as an additive group, $E_n = \{(\eta, \nu) : \eta, \nu \in {}^{\omega_1}(\mathbb{Z}_2)$ are such that $\eta{\restriction}(\omega n) = \nu{\restriction}(\omega n)$ where we interpret $\mathbb{Z}_2$ as the additive group (so $(\mathbb{Z}/2\mathbb{Z}, +, 0)$) and ${}^{\omega_1}(\mathbb{Z}_2)$ as its ω_1-th power as an abelian group.

Definition 1.2. For a complete first order theory T, we say that ψ is a (μ, κ, T)-*candidate* if

1. $\psi \in \mathcal{L}_{\kappa^+,\omega}(\tau_*)$ for some vocabulary $\tau_* \supseteq \tau_T$ of cardinality $\leq \kappa$,

2. $\mathrm{PC}_{\tau(T)}(\psi) \subseteq \mathrm{EC}(T)$,

3. for some $\Phi \in \Upsilon^{\omega\text{-tr}}_\kappa$ satisfying $\tau_\Phi \supseteq \tau_\psi$ and $\mathrm{EM}({}^{\omega \geq}\lambda, \Phi) \models \psi$ for every (equivalent some) λ and Φ witness T is not superstable (for a definition of $\Upsilon^{\omega\text{-tr}}_\kappa$, cf. Definition 3.2).

Recall that by [11, VII]:

Claim 1.3. If a first order complete theory T is not superstable then for some $\Phi \in \Upsilon^{\omega\text{-tr}}_{\tau_2}$ (cf. Definition 3.2) and $\tau_2 \supseteq \tau(\psi)$ of cardinality κ, Φ witnesses that T is not superstable, i.e., for some formulas $\varphi_n(x, \bar{y}_n) \in \mathcal{L}(\tau_T)$, if $I = {}^\omega\lambda$, $M = \mathrm{EM}(I, \Phi)$ then for $\eta \in {}^\omega\lambda$, $n < \omega$ and $\alpha < \lambda$ we have $M \models \varphi_n[\bar{a}_\eta, a_{(\eta\restriction n)^\frown\langle\alpha\rangle}]$ iff $\alpha = \eta(n)$.

Definition 1.4. Fix a structure I.

1. We say that $\langle \bar{a}_t : t \in I \rangle$ is *indiscernible* (in the model $\mathfrak{C}$, over A, if $A = \varnothing$ we may omit it) when ($\bar{a}_t \in {}^{\mathrm{lh}(\bar{a}_t)}\mathfrak{C}$ and) $\mathrm{lh}(\bar{a}_t)$, which is not necessarily finite depends only on the quantifier-free type of t in I and if $n < \omega$ and $\bar{s} = \langle s_0, s_1, \dots, s_{n-1} \rangle, \bar{t} = \langle t_0, \dots, t_{n-1} \rangle$ realize the same quantifier-free type in I then $\bar{a}_{\bar{t}} := \bar{a}_{t_0}{}^\frown \dots {}^\frown \bar{a}_{t_{n-1}}$ and $\bar{a}_{\bar{s}} = \bar{a}_{s_0}{}^\frown \dots {}^\frown \bar{a}_{s_{n-1}}$ realize the same type (over A) in $\mathfrak{C}$.

2. We say that $\langle \bar{a}_u : u \in [I]^{<\aleph_0} \rangle$ is *indiscernible* (in $\mathfrak{C}$, over A) if $n < \omega, w_0, \dots, w_{m-1} \subseteq \{0, \dots, n-1\}$ and $\bar{s} = \langle s_\ell : \ell < n \rangle, \bar{t} = \langle t_\ell : \ell < n \rangle$ realize the same quantifier-free types in I and $u_\ell = \{s_k : k \in w_\ell\}, v_\ell = \{t_k : k \in w_\ell\}$ then $\bar{a}_{u_0}{}^\frown \dots {}^\frown \bar{a}_{u_{n-1}}, \bar{a}_{v_0}{}^\frown \dots {}^\frown \bar{a}_{v_{n-1}}$ realize the same type in $\mathfrak{C}$ (over A).

2 Games, equivalences and questions

We shall define a new notion of equivalence of models below, $\mathrm{EF}^+_{\alpha,\lambda}$-equivalence. Why do we use this particular notion of equivalence? Consider for various γs the game $\mathcal{G}^\gamma_\lambda(M_1, M_2)$ where $M_1, M_2 \in \mathrm{Mod}_T(\lambda)$ and T is a complete first order $\mathcal{L}(\tau)$-theory (cf. Definition 2.3). During a play we can consider dependence relations on "short" sequences from M_ℓ (where $\leq 2^{|\tau|+\aleph_0}$ is the default value), definable in a suitable sense. So if T is a well understood unsuperstable theory like $\mathrm{Th}({}^\omega\omega, E_n)_{n<\omega}$ with $E_n := \{(\eta, \nu) : \eta, \nu \in {}^\omega\omega \text{ and } \eta\restriction n = \nu\restriction n\}$, then even for $\gamma = \omega + 2$ we have $E^+_{\gamma,\lambda}$-equivalence implies being isomorphic. This fits the thesis:

> The desirable dichotomy characterized, on the family of first order T, by the property "$M_1, M_2 \in \mathrm{Mod}_T(\lambda)$ are long game EF-equivalent iff they are isomorphic", is quite similar to the one in [11, XIII]; the structure side is, e.g., T is stable and every $M \in \mathrm{Mod}_T$ is prime over some $\bigcup\{M_\eta : \eta \in I\}$, where $\mathcal{T}$ is a subtree of ${}^{\kappa_r(T)>}\|M\|$ and $\eta \triangleleft \nu \Rightarrow M_\eta \prec M_\nu \prec M, \|M_\eta\| \le 2^{|T|}$ and $\eta \triangleleft \nu \in I$ implies $\mathrm{tp}(M_\nu, \bigcup\{M_\rho : \rho \in T, \rho\restriction(\mathrm{lh}(\nu)+1) \neq \eta\restriction(\mathrm{lh}(\nu)+1))$ does not fork over M_ν, i.e., $\bar{M} = \langle M_\eta : \eta \in \mathcal{T}\rangle$ is a non-forking tree of models with $\le \kappa_r(T)$ many levels.

We think the right (variant of the) question is described as follows. In [11], the original question was about the function $\lambda \mapsto \dot{I}(\lambda, T)$, but the answer is more transparent for the function $\lambda \mapsto \dot{I}\dot{E}(\lambda, T)$. If $\lambda = \mu^+, \mu = \mu^{|T|} = \mathrm{cf}(\mu), T = \mathrm{Th}({}^\omega\omega, E_n)_{n<\omega}$ then (by [2, Th4.4]) for $\gamma \ge \mu\omega$, we get that equivalence implies isomorphism, but not for $\gamma < \mu\omega$; now our Theorem 2.6 is parallel to that. This seems to indicate that $\mathrm{EF}^+_{\gamma,\lambda}$ is suitable for the questions we are asking: it uses the game EF^+, which is more complicated but the length of the game is much "smaller" in the relevant results.

Question 2.1. Classify first order complete T, or at least the countable ones by:

Version $(\mathrm{A})_1$. For every ordinal α, there are a cardinal λ and non-isomorphic $M_1, M_2 \in \mathrm{Mod}_T(\lambda)$ which are $\mathrm{EF}^+_{\alpha,\lambda}$-equivalent (at least, e.g. in some $\mathbf{V}^{\mathbb{P}}, \mathbb{P}$ is $(2^{|T|+|\alpha|})^+$-complete forcing notion).

Version $(\mathrm{A})_0$. Similar version for $\mathrm{EF}_{\alpha,\lambda}$.

Version $(\mathrm{B})_1$. For every cardinal $\kappa > |T|$ and vocabulary $\tau_1 \supseteq \tau_T$ and $\psi \in \mathcal{L}_{\kappa,\omega}(\tau_1)$ such that $\mathrm{PC}_\tau(\psi) \subseteq \mathrm{EC}_T$ has members of arbitrarily large cardinality we have $(a) \Rightarrow (b)$ where

- (a) for every cardinal μ in $\mathrm{PC}_\tau(\psi) := \{M\restriction\tau : M \text{ a model of } \psi\}$ there is a μ-saturated member, and
- (b) for arbitrarily large $\alpha < \lambda$ there are $M_1, M_2 \in \mathrm{PC}_\tau(\psi)$ of cardinality λ with non-isomorphic τ-reducts which are $\mathrm{EF}^+_{\alpha,\lambda}$-equivalent.

Version $(\mathrm{B})_0$. Like $(\mathrm{B})_1$ for $\mathrm{EF}_{\alpha,\lambda}$.

Version $(\mathrm{C})_1$. Like $(\mathrm{B})_1$ using $\psi = \bigwedge T_1$ where $T_1 \supseteq T$ is a first order theory.

Version $(\mathrm{C})_0$. Like $(\mathrm{B})_0$ using $\psi = \bigwedge T_1$ where $T_1 \supseteq T$ is a first order theory.

If the reader is interested to know the reasons to prefer version (B) over version (C), we refer him or her to [17]. Now by the works quoted above, (cf. [3, 3.19] quoted in Theorem 5.1 below), we get that T satisfies $(\mathrm{A})_0$ iff T is a superstable theory with NDOP and OTOP iff $(\mathrm{B})_0$. Of course, if we change the order of the quantifier (to "for arbitrarily large some λ for every $\alpha < \lambda$...") this fails, but we believe solving $(\mathrm{A})_1$ and/or $(\mathrm{B})_1$ will eventually be useful for this case as well. We sum all of this up in the following conjecture.

Conjecture 2.2. For a complete (first order) T the following are equivalent:

(a) for every ordinal α there is some λ and there are non-isomorphic, $\mathrm{EF}^+_{\alpha,\lambda}$-equivalent models $M_1, M_2 \in \mathrm{EC}_T(\lambda)$,

(b) for arbitrarily large λ and for every $\alpha < \lambda$ there are non-isomorphic, $\mathrm{EF}^+_{\alpha,\lambda}$-equivalent models $M_1, M_2 \in \mathrm{EC}_T(\lambda)$,

(c) for every sufficiently large regular λ there are non-isomorphic $M_1, M_2 \in \mathrm{EC}_T(\lambda)$ which are $\mathrm{EF}^+_{\alpha,\lambda}$-equivalent for every $\alpha < \lambda$.

And similarly for "some $T_1 \supseteq T$, $\mathrm{PC}(T_1, T)$ is lean."

We conjecture that we can prove Conjecture 2.2 if we prove that a (countable) fat T is close enough to superstable. Such a result will enable us to generalize proofs in [11, XII] (only now the tree has $\leq \omega_1$ levels rather than ω levels).

Definition 2.3. If M_1, M_2 are models with the same vocabulary and α is an ordinal and μ is a cardinals, and if f is a partial isomorphism from M_1 to M_2, we define the game $\mathcal{G}^\alpha_\mu(f, M_1, M_2)$ between the players ISO, the isomorphism player and AIS, the anti-isomorphism player as follows:

(a) A play lasts α moves.

(b) After β moves a partial isomorphism f_β from M_1 into M_2 is chosen, increasing continuously with β.

(c) In the $(\beta+1)$st move, the player AIS chooses $A_{\beta,1} \subseteq M_1, A_{\beta,2} \subseteq M_2$ such that $|A_{\beta,1}| + |A_{\beta,2}| < 1 + \mu$ and then the player ISO chooses $f_{\beta+1} \supseteq f_\beta$ such that $A_{\beta,1} \subseteq \mathrm{Dom}(f_{\beta+1})$ and $A_{\beta,2} \subseteq \mathrm{Rang}(f_{\beta+1})$.

(d) If $\beta = 0$, ISO chooses $f_0 = f$; if β is a limit ordinal ISO chooses $f_\beta = \bigcup\{f_\gamma : \gamma < \beta\}$.

Player ISO loses if he had no legal move for some $\beta < \alpha$, otherwise he wins the play.

A few notational conventions: Replacing α by $<\alpha$ means "for every $\beta < \alpha$". If μ is 1, we may omit it, and we may write $\leq \mu$ instead of μ^+. Furthermore, if $f = \varnothing$ we may write $\mathcal{G}^\alpha_\mu(M_1, M_2)$.

Based on Definition 2.3, we now say that M_1, M_2 are EF_α-equivalent if the isomorphism player has a winning strategy in the game $\mathcal{G}^\alpha_1(M_1, M_2)$, and M_1, M_2 are $\mathrm{EF}_{\alpha,\mu}$-equivalent or $\mathcal{G}^\alpha_\mu$-equivalent if the isomorphism player has a winning strategy in the game $\mathcal{G}^\alpha_\mu(M_1, M_2)$ defined below.

Let us discuss why we need a more complicated notion EF^+ at all? First, if we like a parallel of [11, XIII], i.e., a game in which set of small cardinality are chosen (e.g., $|T|$ or $2^{|T|}$) rather than just $< \lambda = \|M_\ell\|$, clearly $\mathrm{EF}_{\alpha,\mu}$ cannot help.

Also, consider $\lambda = \mu^+, \mu = \mathrm{cf}(\mu) > |T|$ and an ordinal $\alpha < \lambda$ and ask for which T: "Does $\mathrm{EF}_{\alpha,\lambda}$-equivalence imply isomorphism for any two models M_1, M_2 of T of cardinality λ?" Now we know (by earlier works, cf. Corollary 5.7) for countable T that if $\alpha \in [\omega, \mu \times \omega]$ that the answer (for the pair (α, λ)) is as in the main gap for $\dot{I}\dot{E}$ (T superstable with NDOP and NOTOP). But for larger $\alpha < \lambda$ this is not so, as, e.g., for the prototypical stable unsuperstable T for $\alpha = \mu \times (\omega + 2)$ we get the answer "yes, it is low".

Let us consider the reasons for this. In other words, why do we need $\mu \times (\omega + 2)$ moves, not $(\omega + 2)$ moves? Based on this question, we shall now formulate EF^+. We think that with $\mathrm{EF}^+_{\alpha,\theta,\mu,\lambda}$ for small α, θ, μ and just $\lambda = \|M_\ell\|$ we get the desired dichotomy. In general, we expect the results will be robust under choosing such an exact game; and will resolve the case $\alpha \in (\mu \times (\omega, 2), \lambda)$ above.

More specifically, the reason why $\mathrm{EF}_{\alpha,\lambda}$-equivalence does not imply isomorphism for $M_1, M_2 \in \mathrm{EC}_\lambda(T)$, even in the case $T = \mathrm{Th}({}^\omega\omega, E_n)_{n<\omega}$, is the following: Fix a winning strategy $\mathbf{st}$ for $\mathcal{G}_{\alpha,\lambda}(M_1, M_2)$. If we let $\langle a^\ell_\alpha / E_1^{M_\ell} : \alpha < \lambda \rangle$ list $M_\ell / E_1^{M_\ell}$ and $\mathbf{R} = \{(\alpha, \beta)$: in some short initial segment $\mathbf{x}$ of a play of $\mathcal{G}_{\alpha,\lambda}(M_1, M_2)$ in which the player ISO uses the strategy $\mathbf{st}$, we have $f^{\mathbf{x}}_\alpha(a^1_\alpha) E_1^{M_2} a^2_\beta\}$, we have to find a function h from λ onto λ whose graph is $\subseteq \mathbf{R}$. Now being in a winning position is only enough to show the existence of such h when the game is long enough. For $\mathrm{EF}^+_{\alpha,\theta}$ this changes.

Definition 2.4. We call subsets $\mathbf{R} \subseteq [X]^{<\aleph_0}$ *pre-dependence relations* on X. We say $Y \subseteq X$ is $\mathbf{R}$-*independent* when $[Y]^{<\aleph_0} \cap \mathbf{R} = \varnothing$; of course, an index set with repetitions is considered dependent. We say $\mathbf{R}$ or $(X, \mathbf{R})$ has character $\leq \kappa$ when for every $\mathbf{R}$-independent $Y \subseteq X$ and $\{x\} \subseteq X$ for some $Z \in [Y]^{<\kappa}$ the set $(Y \backslash Z) \cup \{x\}$ is $\mathbf{R}$-independent. We say that $\mathbf{R}$ is a

k-dependence relation on X (if $k = 1$ we may omit it) when $\mathbf{R}$ is a subset of $[X]^{<\aleph_0}$, if $k = 0$ then $\mathbf{R} = [X]^{<\aleph_0}$, if $k = 2$ then $\mathbf{R}$-independence satisfies the exchange principle (so dimension is well defined, as for regular types), and if $k = 1$ then $\mathbf{R}$ is trivial. We say $\mathbf{R}$ is *trivial* when for every $Y \subseteq X, Y$ is $\mathbf{R}$-independent iff every $Z \subseteq [Y]^{\leq 2}$, is $\mathbf{R}$-independent. For a 1-dependence relation $\mathbf{R}$, let $E_{\mathbf{R}} = \{\{x_1, x_2\} : x_1 = x_2 \in X$ or $\{x_1\}, \{x_2\} \in \mathbf{R}$ or $\{x_1, x_2\} \in \mathbf{R} \wedge \{x_1\} \notin \mathbf{R} \wedge \{x_2\} \notin \mathbf{R}\}$ is an equivalence relation on X; pedantically we should write $E_{X,\mathbf{R}}$.

Definition 2.5. For an ordinal γ, cardinals $\theta \leq \mu$, vocabulary τ and τ-models M_1, M_2 and partial isomorphism f from M_1 to M_2, we define a game $\mathcal{G}^{+,k}_{\gamma,\theta,\mu,\lambda}(f, M_1, M_2)$, between the player ISO (isomorphism) and AIS (anti-isomorphism).

A play last γ moves; in the βth move a partial isomorphism f_β from M_1 to M_2 is chosen by ISO, extending f_α for $\alpha < \beta$ such that $f_0 = f$ and for limit β we have $f_\beta = \bigcup\{f_\alpha : \alpha < \beta\}$ and for every $\beta < \alpha$ the set $\mathrm{Dom}(f_{\beta+1}) \backslash \mathrm{Dom}(f_\beta)$ has cardinality $< 1 + \mu$; let f^ℓ_β be f_β if $\ell = 1, f^{-1}_\beta$ if $\ell = 2$. During a play, the player ISO loses if he has no legal move and he wins in the end of the play iff he always had a legal move. In the $(\beta + 1)$st move, the AIS player does one of the following cases:

Case 1. The AIS player chooses $A_\ell = A^\ell_\beta \subseteq M_\ell$ for $\ell = 1, 2$ such that $|A_1| + |A_2| < 1 + \mu$ and then ISO chooses f_β as above such that $A_\ell \subseteq \mathrm{Dom}(f^\ell_\beta)$ for $\ell = 1, 2$.

Case 2. First the AIS player chooses[4] a pre-dependence relation $\mathbf{R}_\ell$ on ${}^{\theta>}(M_\ell)$ and $\mathcal{A}_\ell \subseteq {}^{\theta>}(M_\ell)$ of cardinality $\leq \lambda$ for $\ell = 1, 2$ such that:

(a) if $k = 0$ then $\mathbf{R}_\ell = [{}^{\theta>}(M_\ell)]^{<\aleph_0}$, so really an empty case

(b) if $k = 1, 2$ then $\mathbf{R}_\ell$ is a k-dependence relation

(c) if $k = 1, 2$ and $\ell = 1, 2$ and $n < \omega$ and $\bar{a}_0, \ldots, \bar{a}_{n-1} \in {}^{\theta>}(M_\ell)$ then the truth value of $\{\bar{a}_0, \ldots, \bar{a}_{n-1}\} \in \mathbf{R}_\ell$ depends just on the complete first order type which $\langle \bar{a}_0, \ldots, \bar{a}_{n-1} \rangle$ realizes on $\mathrm{Dom}(f^\ell_\beta)$ inside the model M_ℓ.

After that, player ISO does one of the following:

Subcase 2A. First, assume $k = 2$. The player ISO chooses $\langle (\bar{a}^1_\zeta, \bar{a}^2_\zeta) : \zeta < \lambda \rangle$ such that for $\ell = 1, 2$:

(α) for each $\zeta < \lambda$ there is some $\varepsilon < \theta$ such that $\bar{a}^\ell_\zeta \in {}^{\varepsilon}(M_\ell)$

[4] Note that for $k = 0, 1$ we require "$\mathcal{L}(\tau_T)$-definable $\mathbf{R}_\ell$ such that f maps the definition of $\mathbf{R}_1$ to the one of $\mathbf{R}_2$"; moreover we expect that we can demand it is as in the case of using regular types.

(β) $\langle \bar{a}^\ell_\zeta : \zeta < \lambda \rangle$ is independent for $\mathbf{R}_\ell$

(γ) each $\bar{a} \in \mathcal{A}_\ell$ does $\mathbf{R}_\ell$-depend on $\{\bar{a}^\ell_\zeta : \zeta < \lambda\}$.

Then AIS chooses $\zeta < \lambda$ and ISO chooses $f_{\beta+1} \supseteq f_\beta$ such that $f_\beta(\bar{a}^1_\zeta) = \bar{a}^2_\zeta$.

Now assume $k = 1$. Then player ISO chooses equivalence relations E_ℓ on ${}^{\theta>}(M_\ell)$ which refine the relations $E_{\mathbf{R}_\ell}$ (cf. Definition 2.4) and equality of length, and chooses a function h from the family of E_1-equivalence classes onto the family of E_2-equivalence classes which preserve cardinality up to λ; that is, if $h(\bar{a}_1/E_1) = \bar{a}_2/E_2$ then $\mathrm{lh}(\bar{a}_1) = \mathrm{lh}(\bar{a}_2)$,$\min\{\dim(\bar{a}_1/E_1), \lambda) = \min\{\dim(\bar{a}_2/E_2), \lambda\}$.

Then the AIS player chooses a pair $(\bar{a}_1, \bar{a}_2)$ such that $\bar{a}_\ell \in {}^{\theta>}(M_\ell)$ for $\ell = 1, 2$ such that $h(\bar{a}_1/E_1) = (\bar{a}_2/E_2)$ and ISO has to choose $f_{\beta+1} \supseteq f_\beta$ such that $f(\bar{a}_1) = \bar{a}_2$.

Subcase 2B. The player ISO chooses $f_{\beta+1} \supseteq f_\beta$ as required such that for some $n < \omega$ and $\bar{a}^1_\ell \in {}^\varepsilon\mathrm{Dom}(f_\beta)$ for $\ell \leq n$ we have: $\{\bar{a}^1_0, \ldots, \bar{a}^1_{n-1}\}$ is $\mathbf{R}_1$-dependent iff $\{f_\beta(\bar{a}^1_0), \ldots, f_\beta(\bar{a}^1_{n-1})\}$ is not $\mathbf{R}_2$-dependent.

Based on Definition 2.5, we say that models M_1, M_2 are $\mathrm{EF}^{+,k}_{\gamma,\theta,\mu,\lambda}$-equivalent ($k \in \{0, 1, 2\}$) if the player ISO has a winning strategy in the game $\mathcal{G}^k_{\gamma,\theta,\mu,\lambda}(M_1, M_2)$. In this, we always assume $\aleph_0 \leq \theta \leq \mu$. If $\mu = \min\{\|M_1\|, \|M_2\|\}$, $k = 1$, or $\theta = (2^{|\tau(M_\ell)|+\aleph_0})^+$, we may omit these parameters.

Theorem 2.6. Let $T = \mathrm{Th}({}^\omega\omega, E_n)_{n<\omega}$ with $E_n = \{(\eta, \nu) : \eta \in {}^\omega\omega, \nu \in {}^\omega\omega$ and $\eta{\restriction}n = \nu{\restriction}n\}$. Then if M_1 and M_2 are models of T of cardinality λ that are $\mathrm{EF}^+_{\omega+2,\aleph_0,\aleph_0,\lambda}$-equivalent, they are isomorphic.

Proof. We choose a winning strategy $\mathbf{st}$ of the isomorphism player in the game $\mathcal{G}_{\omega+2,\aleph_0,\aleph_0,\lambda}(M_1, M_2)$.

Step A. By the choice of T for $\ell = 1, 2$ we can find $\mathcal{T}_\ell, \bar{\mathbf{a}}_\ell$ such that $\mathcal{T}_\ell$ is a subtree of ${}^{\omega>}\lambda$, $\bar{\mathbf{a}}_\ell = \langle a^\ell_\eta : \eta \in \mathcal{T}_\ell \rangle$, $a^\ell_\eta \in M_\ell$, and if $\eta \in \mathcal{T}_\ell$ and $\mathrm{lh}(\eta) = n$ then $\langle a^\ell_\nu / E^{M_\ell}_{n+1} : \nu \in \mathrm{suc}_{\mathcal{T}_\ell}(\eta) \rangle$ lists $\{b/E^{M_\ell}_{n+1} : b \in M_\ell$ and $b \in a^\ell_\eta / E^{M_\ell}_n\}$ without repetitions. Let $\mathcal{T}_{\ell,n} = \{\eta \in \mathcal{T}_\ell : \mathrm{lh}(\eta) = n\}$ and let $\mathcal{T}_{\ell,\omega} = \{\eta \in {}^\omega\lambda : \eta{\restriction}n \in \mathcal{T}_\ell$ for every $n < \omega\}$. Lastly, let $\bar{\mu}_\ell = \langle \mu^\ell_\eta : \eta \in \mathcal{T}_{\ell,\omega} \rangle$, where

$$\mu^\ell_\eta = |\{b \in M_\ell : b \in a^\ell_{\eta{\restriction}n} / E^{M_\ell} \text{ for every } n < \omega\}|.$$

Step B. Clearly M_1, M_2 are isomorphic if and only if there is an isomorphism h from $\mathcal{T}_1$ onto $\mathcal{T}_2$ (i.e., h maps $\mathcal{T}_{1,n}$ onto $\mathcal{T}_{2,n}$, h preserves the length, $\eta \triangleleft \nu$ and $\eta \ntriangleleft \nu$) such that letting $h_n = h{\restriction}\mathcal{T}_{1,n}$ and h_ω be the mapping from $\mathcal{T}_{1,\omega}$ onto $\mathcal{T}_{2,\omega}$ which h induces (so $h_\omega(\eta) = \bigcup_{n<\omega} h_n(\eta{\restriction}n)$) we have that $\eta \in \mathcal{T}_{1,\omega}$ implies $\mu^1_\eta = \mu^2_{h(\eta)}$.

STEP C. By induction on n we choose $h_n, \bar{\mathbf{x}}_n$ such that h_n is a one-to-one mapping from $\mathcal{T}_{1,n}$ onto $\mathcal{T}_{2,n}$, if $m < n$ and $\eta \in \mathcal{T}_{1,n}$ then $h_m(\eta\restriction m) = (h_n(\eta))\restriction m$, $\bar{\mathbf{x}}_n = \langle \mathbf{x}^n_\eta : \eta \in \mathcal{T}_{1,n}\rangle$, and

1. (a) $\mathbf{x}^n_\eta$ is an initial segment of a play of the game $\eth_{\omega+2,\aleph_0,\aleph_0,\lambda}(M_1, M_2)$,
   (b) in $\mathbf{x}^n_\eta$ only finitely many moves have been played (can specify), the last one is $m(\mathbf{x}^n_\eta)$,
   (c) in $\mathbf{x}^n_\eta$, the player ISO uses his winning strategy $\mathbf{st}$.
2. If $\eta_1 \in \mathcal{T}_{1,n}$ and $\eta_2 = h_n(\eta_1)$ then for some $b_1 \in \mathrm{Dom}(f^{\mathbf{x}^n_\eta}_{m(\mathbf{x}^n_\eta)})$ we have $b_1 \in a^1_\eta/E^{M_1}_n$ and $f^{\mathbf{x}^n_\eta}_{m(\mathbf{x}^n_\eta)}(b_1) \in a^2_{h_n(\eta)}/E^{M_2}_n$.
3. If $\nu \triangleleft \eta \in \mathcal{T}_{1,n}$ then $\mathbf{x}^{\mathrm{lh}(\nu)}_\nu$ is an initial segment of $\mathbf{x}^n_\eta$.

Why does the induction work? Note that h_0 is uniquely determined. As for $\mathbf{x}^0_\varnothing$, any $\mathbf{x}$ as in 1. is fine, as long as at least one move was done (note that $E^{M_\ell}_0$ has one and only one equivalence class). In the successor step, $n = m+1$, $h_m, \bar{\mathbf{x}}_m$ has been chosen. Let $\eta_1 \in \mathcal{T}_{1,m}$ and let $\eta_2 = h_m(\eta_1)$ and $\mathbf{F}_{\eta_1} := \{(\nu_1, \nu_2) : \nu_1 \in \mathrm{suc}_{\mathcal{T}_1}(\eta_1), \nu_2 \in \mathrm{suc}_{\mathcal{T}_2}(\eta_2)$ and there is $\mathbf{x}$ as in 1. such that $\mathbf{x}^m_{\eta_1}$ is an initial segment of $\mathbf{x}$ and for some $b_1 \in \mathrm{Dom}(f^{\mathbf{x}}_{m(\mathbf{x})})$ we have $b_1 \in a^1_{\nu_1}/E^{M_1}_n$ and $f^{\mathbf{x}}(b_1) \in a^2_{\nu_2}/E^{M_2}_n\}$. Now to do the induction step, it suffices to prove that: if $\eta_1 \in \mathcal{T}_{1,m}$ then there is a one-to-one function h_{n,η_1} from $\mathrm{suc}_{\mathcal{T}_1}(\eta_1)$ onto $\mathrm{suc}_{\mathcal{T}_2}(\eta_2)$ such that $\nu \in \mathrm{suc}_{\mathcal{T}_1}(\eta_1) \Rightarrow (\nu, h_{n,\eta_1}(\nu)) \in \mathbf{F}_{\eta_1}$. However by **Case 2** in Definition 2.5 this holds.

STEP D. So we can find $\langle h_n : n < \omega\rangle, \langle \mathbf{x}_\eta : \eta \in \mathcal{T}_1\rangle$ as in STEP C. Let $h := \bigcup\{h_n : n < \omega\}$, clearly it is an isomorphism from $\mathcal{T}_1$ onto $\mathcal{T}_2$ and h_ω is well defined (cf. STEP B). So it is enough to check the sufficient condition for $M_1 \cong M_2$ there, i.e., $\eta \in \mathcal{T}_{1,\omega} \Rightarrow \mu_{1,\eta} = \mu_{2,h_\omega(\eta)}$. But if $\eta \in \mathcal{T}_{1,\omega}$ then $\langle \mathbf{x}_{\eta\restriction n} : n < \omega\rangle$ is a sequence of initial segments of a play of $\mathcal{G}$ with ISO using his winning strategy $\mathbf{st}$, increasing with n, each with finitely many moves. So $\mathbf{x}_\eta$, defined as the limit $\langle \mathbf{x}_{\eta\restriction n} : n < \omega\rangle$, is an initial segment of the play $\mathcal{G}$, with $\leq \omega$ moves and $f^{\mathbf{x}_\eta}_{\mathbf{m}(\mathbf{x}_\eta)} = \bigcup\{f^{\mathbf{x}_{\eta\restriction n}}_{m(\mathbf{x}_{\eta\restriction n})} : n < \omega\}$.

Clearly if $n < \omega$, then $f(a^1_{\eta\restriction n})E^{M_2}_n a^2_{h_n(\eta\restriction n)}$. As we have one move left and can use **Case 2** in Definition 2.5 we are done. Q.E.D.

The following claim says that the games in Definitions 2.3 and 2.5 are equivalent, i.e., the ISO player wins one iff he wins the other (when $\lambda = \mu^+, \alpha < \lambda$ divisible enough).

Theorem 2.7. Let M_1, M_2 are τ-models, $\lambda = \lambda_1^+, \lambda_1 \geq \mu$ and $\theta \leq \mu \leq \lambda$ and $\gamma \leq \mu$ and $\mathrm{cf}(\mu) < \mu \Rightarrow \lambda_1 > \mu$ and $\lambda \in \check{I}[\lambda]$. Define $\gamma(*) := \lambda_1 \times \gamma$.

1. Assume that M_1, M_2 are $\mathrm{EF}_{\gamma(*),\mu}$-equivalent and that $\|M_\ell\| = \lambda = \lambda^{<\theta}$ for $\ell = 1, 2$. Then M_1, M_2 are $\mathrm{EF}^+_{\gamma,\theta,\mu,\lambda}$-equivalent.

2. If M_1, M_2 are $\mathrm{EF}^+_{\gamma,\theta,\mu,\lambda}$-equivalent, they are $\mathrm{EF}_{\gamma,\mu}$-equivalent.

3. If $\gamma_1 \leq \gamma_2, \theta_1 \leq \theta_2, \mu_1 \leq \mu_2, \lambda_1 \leq \lambda_2$ and M_1, M_2 are $\mathrm{EF}^+_{\gamma_2,\theta_2,\mu_2,\lambda_2}$-equivalent, they are $\mathrm{EF}^+_{\gamma_1,\theta_1,\mu_1,\lambda_1}$-equivalent.

Proof. To show 1., let us assume $\gamma(*) = \lambda_1 \times \lambda_1 \times \gamma$. Let $\mathbf{st}$ be a winning strategy of the player ISO in the game $\mathcal{G}^{\gamma(*)}_\mu$. We try to use it as a winning strategy of the ISO player in the game $\mathcal{G}_{\gamma,\theta,\mu,\lambda}(M_1, M_0)$. Well, the $f^{\mathbf{x}}_\alpha$ may have too large a domain, so in the βth move ISO plays as auxiliary moves $\mathbf{x}_\beta$ for $\mathcal{G}_{\gamma,\theta,\mu,\lambda}$ and $A^1_\beta \subseteq \mathrm{Dom}(f^{\mathbf{x}_\beta})$ of cardinality $< \mu$ (or $\leq \mu$ if $\mu > \mathrm{cf}(\mu) \wedge \beta \geq \mathrm{cf}(\mu)$) and he actually plays $f^{\mathbf{x}_\beta} {\restriction} A^1_\beta$, i.e., is an initial segment of a play of $\mathcal{G}_{\gamma,\mu}$ of length β in which the ISO player uses the strategy $\mathbf{st}$ such that $[\beta_1 < \beta \Rightarrow \mathbf{x}_{\beta_1}$ is an initial segment of $\mathbf{x}_\beta]$.

The only problem is when $\beta = \alpha + 1$ and if **Case 2** of Definition 2.5 occurs, i.e., with player AIS choosing $\mathbf{R}^1_\beta, \mathbf{R}^2_\beta$. We may for notational simplicity choose $\varepsilon < \theta$ and deal only with $A_\ell \cap {}^\varepsilon(M_\ell)$ for $\ell = 1, 2$.

We can consider $\mathbf{x}_\beta$ extending $\mathbf{x}_\alpha$; if it is as required in **Subcase 2B** of Definition 2.5 we are done. Let $\mathbf{F}^1_\beta := \{(\bar a_1, \bar a_2)$: for some $\varepsilon < \theta$, $\bar a_\ell \in {}^\varepsilon(M_\ell)$ for $\ell = 1, 2$ and there is a candidate $\mathbf{x}_\beta$ for the βth move such that $f^{\mathbf{x}_\beta}(\bar a_1) = \bar a_2\}$. Let $\mathbf{F}^2_\beta = \{(\bar a_2, \bar a_1) : (\bar a_1, \bar a_2) \in \mathbf{F}^1_\beta\}$, $\mathcal{A}^1_\ell = \mathcal{A}_\ell$, $\mathcal{A}^2_\ell = \{\bar a \in \mathcal{A}_\ell$: the number of $\bar b$ such that $(\bar a, \bar b) \in \mathbf{F}^\ell_\beta$ is $\leq \lambda\}$, and $\mathcal{A}^3_\ell = \mathcal{A}^2_\ell \cup \{\bar a$: for some $\bar b \in \mathcal{A}^2_{3-\ell}$ we have $(\bar a, \bar b) \in \mathbf{F}^\ell_\beta\}$. So $|\mathcal{A}^3_\ell| \leq \lambda$ by the assumption and let $\langle \bar a^\ell_\zeta : \zeta < \lambda\rangle$ list $\mathcal{A}^3_\ell$ possibly with repetitions. Then by the basic properties of dependence relations, it is enough to take care of $\mathcal{A}^3_\ell \cap \mathcal{A}_\ell$ for $\ell = 1, 2$. So we can continue.

Let S be the set of limit ordinals $\delta < \lambda$ such that: for a club of $\delta_* \in [\delta, \lambda)$ of cofinality $\aleph_0$ we can find $\langle \bar b^\ell_\zeta : \zeta \in [\delta, \delta_*)\rangle$ for $\ell = 1, 2$ such that $\bar b^\ell_\zeta \in \{\bar a^\ell_\xi : \xi \in [\delta, \delta_*)\}$, $(\bar b^1_\zeta, \bar b^2_\zeta) \in \mathbf{F}^1_\beta$, $\langle \bar b^\ell_\zeta : \zeta \in [\delta, \delta_*)\rangle$ is $\mathbf{R}_\ell$-independent over $\{\bar a^\ell_\zeta : \zeta < \delta\}$, and if $\zeta < \delta_*$ and $\bar a^\ell_\zeta \in \mathcal{A}_\ell$ then $\bar a^\ell_\zeta$ does R_ℓ-depend on $\{\bar a^\ell_\zeta : \zeta < \delta\} \cup \{\bar b^\ell_\zeta : \zeta \in [\delta, \delta_*)\}$.

If S is not stationary we can easily finish (we start by playing ω moves in $\mathcal{G}^\gamma_\mu$). So assume S is stationary, hence for some regular $\sigma \leq \lambda_1$ the set $S' = \{\delta \in S : \mathrm{cf}(\delta) = \sigma\}$ is stationary. By playing $\sigma + \omega$ moves (recalling $\lambda \in \check I[\lambda]$) we get a contradiction to the definition of S.

Claims 2. and 3. are obvious. Q.E.D.

In Theorem 2.7, Claim 1., to get the exact $\gamma(*)$, we combine partial isomorphisms. So we simulate two plays and use the composition of the $f^{\mathbf{x}^i_\beta}$'s from two plays where each player ISO uses a winning strategy $\mathbf{st}$.

Proposition 2.8. Define a variant of Definition 2.5 as follows: in **Case 2** use a dependence relation $\mathbf{R}_\ell$ on $\kappa \times {}^{\theta>}(M_\ell)$ (or equivalently $C \times {}^{\theta>}(M_\ell)$ for a set C of cardinality $\leq \kappa$). If $\kappa \leq 2^{<\theta}$ we get an equivalent game.

In Proposition 2.8, we can replace $2^{<\theta}$ by a larger cardinal for "interesting" cases of M_1, M_2.

Proof. Without loss of generality, $\|M_\ell\| > 1$, now let $\langle \eta_\alpha : \alpha < \kappa \rangle$ be a sequence of pairwise distinct members of ${}^{\kappa>}2$. now we define $F_\ell : {}^{\theta>}(M_\ell) \to {}^{\theta>}(M_\ell)$ as follows: for $\bar{a} \in {}^{\theta>}(M_\ell)$ let $\mathbf{i}(\bar{a}) = \min\{i : 2i \geq \mathrm{lh}(\bar{a}) \text{ or } 2i+1 < \mathrm{lh}(\bar{a}) \wedge a_{2i} \neq a_{2i+1}\}$ and $\eta_{\bar{a}} = \langle \mathrm{tv}(a_{2\mathbf{i}(\bar{a})+2+2j} = a_{2\mathbf{i}(\bar{a})+2+2j+1}) : j \geq 0 \text{ and } 2\mathbf{i}(\bar{a}) + 2 + 2j + 1 < \mathrm{lh}(\bar{a}) \rangle$ (where tv stands for "truth value"), and $\alpha(\bar{a}) = \mathrm{Min}\{\alpha \leq \kappa : \text{ if } \alpha < \kappa \text{ then } \eta_\alpha = \eta_{\bar{a}}\}$. Finally, let $F_\ell(\bar{a})$ be $(\alpha(\bar{a}), \langle a_{2j} : j < i(\bar{a})\rangle)$ if $\mathbf{i}(a) < \mathrm{lh}(\bar{a}) \wedge \alpha(\bar{a}) < \kappa$, and $(0, \bar{a})$ otherwise. Define $\mathbf{R}'_\ell := \{\mathcal{A} : \mathcal{A} \subseteq {}^{\theta>}(M_\ell)$ and $\{F(\bar{a}) : \bar{a} \in \mathcal{A}\} \in \mathcal{R}$ or for some $\bar{a}' \neq \bar{a}'' \in \mathcal{A}$ we have $F(\bar{a}') = \bar{a}''\}$. Now check. Q.E.D.

Proposition 2.9. Let K be a class of τ_0-structures and $\Phi \in \Upsilon[K]$ (cf. Definition 3.2), used here for the class $K = K_{\mathrm{or}}$ of linear orders and K_{oi} (cf. Definition 3.1). Suppose that the structures I_1, I_2 are $\mathrm{EF}^+_{\gamma,\theta,\mu,\lambda}$-equivalent, that $M_\ell = \mathrm{EM}_\tau(I_\ell, \Phi)$ for $\ell = 1, 2$ for some $\tau \subseteq \tau_\Phi$, and that $\mu \geq \aleph_0$ and $|\tau_\Phi| < \theta$. Then M_1, M_2 are $\mathrm{EF}^+_{\gamma,\theta,\mu,\lambda}$-equivalent.

Proof. Let **st** be a winning strategy of player ISO in the game $\mathcal{G}^+_{\gamma,\theta,\mu,\lambda}(I_1, I_2)$. We define a strategy $\mathbf{st}_*$ of player ISO in the game $\mathcal{G}^+_{\gamma,\theta,\mu,\lambda}(M_1, M_2)$ as follows.

During a play of it after β moves a partial isomorphism f^*_α from M_1 to M_2 has been chosen, but player ISO also simulates a play of $\mathcal{G}^+_{\gamma,\theta,\mu,\lambda}(I_1, I_2)$ in which we call the function h_α, and in which he uses the winning strategy **st** and $f_\alpha \subseteq \hat{h}_\alpha$ where $\hat{h}_\alpha$ is defined by

$$\hat{h}_\alpha(\sigma^{M_1}(a_{t_0}, \ldots, a_{t_{n-1}})) = \sigma^{M_\ell}(a_{h_\alpha(t_0)}, \ldots, a_{h_\alpha(t_{n-1})})$$

for $n < \omega$, a term $\sigma(x_0, \ldots, x_{n-1})$ of τ_Φ, and $t_0, \ldots, t_{n-1} \in \mathrm{Dom}(h_\alpha)$.

Why can ISO follow this strategy $\mathbf{st}_*$? Suppose we arrive in the βth move. The point to check is **Case 2** in Definition 2.5, so the AIS player has chosen $\mathbf{R}_1, \mathbf{R}_2, \mathcal{A}_1, \mathcal{A}_2$ as there. Let $\{\bar{\sigma}_\zeta(\bar{x}_\zeta) : \zeta < 2^{<\theta}\rangle$ list $\{\bar{\sigma}(\bar{x}) : \bar{\sigma}(\bar{x}) = \langle \sigma_i(\bar{x}) : i < \mathrm{lh}(\bar{\sigma})\rangle, \mathrm{lh}(\bar{\sigma}) < \theta, \mathrm{lh}(\bar{x}) < \theta$ and each σ_i is a τ_K-term. Clearly ${}^{\theta>}(M_\ell) = \{\bar{\sigma}^{M_\ell}_\zeta(\bar{t}) : \zeta < 2^{<\theta}$ and $\bar{t} \in {}^{\mathrm{lh}(\bar{x}_\zeta)}(I_\ell)\}$, so by Proposition 2.8, we can assume "$\mathbf{R}_\ell$ is a dependence relation on $\{(\zeta, \bar{t}_\zeta) : \zeta < 2^{<\theta}, \bar{t}_\zeta \in {}^{\theta}(I_\theta)$ and $\mathrm{lh}(\bar{t}) = \mathrm{lh}(\bar{t}_\zeta)\}$". I.e., $\mathbf{R}'_\ell = \{u : \{\sigma^{M_\ell}_\zeta(\bar{t}) : (\zeta, \bar{t}) \in u\} \in \mathbf{R}_\ell$ or there are $(\zeta_1, \bar{t}_1) \neq (\zeta_2, \bar{t}_2)$ from u such that $\sigma^{M_\ell}_{\zeta_1}(\bar{a}_{\bar{t}_1}) = \sigma^{M_\ell}_{\zeta_2}(\bar{a}_{\bar{t}_2})\}$. The rest is clear. Q.E.D.

3 The properties of T and relevant indiscernibility

In [11, VIII], [15, VI] we use as indiscernible sets trees with $\omega + 1$ levels, suitable for dealing with unsuperstable (complete first order) theories. Here instead we use a linear order and family of ω-sequences from it, suitable for our case. In the following, the letters "oi" stands for "order" and "increasing" (ω-sequences).

Definition 3.1. We let K^{oi}_λ be the class of structures $\mathbf{J}$ of the form

$$(J, Q, P <, F_n)_{n<\omega} = (|\mathbf{J}|, P^{\mathbf{J}}, Q^{\mathbf{J}}, <^{\mathbf{J}}, F^{\mathbf{J}}_n),$$

where $J = |\mathbf{J}|$ is a set of cardinality $\lambda, <^{\mathbf{J}}$ a linear order on $Q^{\mathbf{J}} \subseteq J, P^{\mathbf{J}} = |\mathbf{J}| \setminus Q^{\mathbf{J}}, F^{\mathbf{J}}_n$ a unary function, $F^{\mathbf{J}}_n {\restriction} Q^{\mathbf{J}} =$ the identity and $a \in J \setminus Q^{\mathbf{I}}$ implies $F^{\mathbf{J}}_n(a) \in Q^{\mathbf{J}}$, $n \neq m$ implies $F^{\mathbf{J}}_n(a) \neq F^{\mathbf{J}}_m(a)$, and for simplicity $a \neq b \in P^M$ implies $\bigvee_{n<\omega} F_n(a) \neq F_n(b)$; lastly, we add that $n < m$ implies $F^{\mathbf{J}}_n(a) <^{\mathbf{J}} F^{\mathbf{J}}_m(a)$ (there is a small price). We stipulate $F^{\mathbf{J}}_\omega =$ the identity on $|\mathbf{J}|$ and $I^{\mathbf{J}} = (Q^{\mathbf{J}}, <^{\mathbf{J}})$. We write $K_{\mathrm{oi}} := \bigcup\{K^{\mathrm{oi}}_\lambda : \lambda \text{ a cardinal}\}$.

For a linear order I and $\mathfrak{S} \subseteq \mathrm{incr}_\omega(I)$ (cf. Definition 1.4), we let $\mathbf{J} = \mathbf{J}_{I,\mathfrak{S}}$ be the *derived member* of K_{oi} which means: $|\mathbf{J}| = I \cup \mathfrak{S}, (Q^{|\mathbf{J}|}, <^{\mathbf{J}}) = I, F^{\mathbf{J}}_n(\eta) = \eta(n)$ for $n < \omega, F^{\mathbf{J}}_n(t) = t$ for $t \in I$.

Finally, K^{or}_λ is the class of linear order of cardinality λ and $K_{\mathrm{or}} = \bigcup\{K^{\mathrm{or}}_\lambda : \lambda \text{ a cardinal}\}$.

Definition 3.2. For a vocabulary τ_1 let $\Upsilon^{\mathrm{oi}}_{\tau_1}$ be the class of functions Φ with domain $\{\mathrm{tp}_{\mathrm{qf}}(\bar{t}, \varnothing, \mathbf{J}) : \bar{t} \in {}^{\omega>}|\mathbf{J}|, \mathbf{J} \in K^{\mathrm{oi}}\}$ and if $q(x_0, \ldots, s_{m-1}) \in \mathrm{Dom}(\Phi)$ then $\Phi(q)$ is a complete quantifier free n-type in $\mathcal{L}(\tau_1)$ with the natural compatibility functions.

Let $\Upsilon^{\mathrm{oi}}_\kappa = \{\Phi : \Phi \in \Upsilon^{\mathrm{oi}}_{\tau_1}$ for some vocabulary τ_1 of cardinality $\kappa\}$. For $\Phi \in \Upsilon^{\mathrm{oi}}_\kappa$ let $\tau(\Phi) = \tau_\Phi$ be the vocabulary τ_1 such that $\Phi \in \Upsilon^{\mathrm{oi}}_{\tau_1}$.

For $\Phi \in \Upsilon^{\mathrm{oi}}_\kappa$ and $\mathbf{J} \in K^{\mathrm{oi}}$ let $\mathrm{EM}(\mathbf{J}, \Phi)$ be "the" τ_Φ-model M_1 generated by $\{a_t : t \in \mathbf{J}\}$ such that

$$\mathrm{tp}_{\mathrm{qf}}(\langle a_{t_0}, \ldots, a_{t_{n-1}}\rangle, \varnothing, M_1) = \Phi(\mathrm{tp}_{\mathrm{qf}}(\langle t_0, \ldots, t_{n-1}\rangle, \varnothing, \mathbf{J})$$

for all $n < \omega$ and $\bar{t} \in {}^n\mathbf{J}$. If $\tau \subseteq \tau_\Phi$ then $\mathrm{EM}_\tau(\mathbf{J}, \Phi)$ is the τ-reduct of $\mathrm{EM}(J, \Phi)$.

In the above, we can replace K_{oi} by any class K of τ_K-structures. For instance, for $K = K_{\mathrm{or}}$, the class of linear orders, we get $\Upsilon^{\mathrm{or}}_{\tau_1}$, $\Upsilon^{\mathrm{or}}_\kappa$ and $\mathrm{EM}(I, \Phi)$, $\mathrm{EM}_\tau(I, \Phi)$; for $K = K^{\omega\text{-tr}}$, the class of trees with $\omega + 1$ levels (with a linear order on the successor of any member of level $< \omega$), we get $\Upsilon^{\omega\text{-tr}}_{\tau_1}$, $\Upsilon^{\omega\text{-tr}}_\kappa$ and $\mathrm{EM}(I, \Phi)$, $\mathrm{EM}_\tau(I, \Phi)$.

The following definitions are from from [6, §3]; cf. [7, §1].

Definition 3.3. A (complete first order) T is $\aleph_0$-*independent* (or, *not strongly dependent*) if there is a sequence $\bar{\varphi} = \langle \varphi_n(x, \bar{y}_n) : n < \omega \rangle$, of (first order) formulas[5] such that T is consistent with Γ_λ for some (equivalently, every) $\lambda \geq \aleph_0$ where

$$\Gamma_\lambda = \{\varphi_n(x_\eta, \bar{y}^n_\alpha)^{\mathrm{if}(\alpha=\eta(n))} : \eta \in {}^\omega\lambda, \alpha < \lambda, n < \omega\}.$$

A theory T is *strongly stable* if it is stable and strongly dependent.

Proposition 3.4. If T is a complete first order theory which is not strongly dependent, and $T_1 \supseteq T$ is another complete first order theory (without loss of generality with Skolem functions), then we can find $\bar{\varphi} = \langle \varphi_n(x, \bar{y}_n) : n < \omega \rangle, \bar{y}_n \trianglelefteq \bar{y}_{n+1}$ and $\varphi_n(x, \bar{y}_n) \in \mathcal{L}(\tau_T)$ for $n < \omega$ such that for any $\mathbf{J} \in K_{\mathrm{oi}}$ we can find $M, \langle \bar{a}_t : t \in \mathbf{J} \rangle$ such that

- M is the Skolem hull of $\{\bar{a}_t : t \in \mathbf{J}\}$,
- $\bar{a}_t \in {}^\omega M$ for $t \in I^{\mathbf{J}}, \bar{a}_\eta = \langle a_\eta \rangle \in M_1$ for $\eta \in P^{\mathbf{J}}$,
- for $\eta \in P^{\mathbf{J}}, t \in Q^{\mathbf{J}}$ and $n < \omega$ we have $M \models \varphi_n[\bar{a}_\eta, \bar{a}_t]$ iff $F_n(\eta) = t$; (pedantically we should write $\varphi_n(a_\eta, \bar{a}_t {\restriction} \mathrm{lh}(\bar{y}_n))$),
- $\langle \bar{a}_t : t \in \mathbf{J} \rangle$ is indiscernible in M for the index model $\mathbf{J}$,
- M is a model of T_1, and
- in fact (not actually used, cf. Definition 3.2) there is $\Phi \in \Upsilon^{\mathrm{oi}}_{|T_1|}$ depending on $T_1, \bar{\varphi}$ only such that $M = \mathrm{EM}(\mathbf{J}, \Phi)$, in fact if $n < \omega, \bar{t} = \langle t_\ell : \ell < n \rangle \in \mathbf{J}$ then $\mathrm{tp}_{\mathrm{qf}}(\bar{a}_{t_0}{}^\frown \ldots {}^\frown \bar{a}_{t_{n-1}}, \varnothing, M) = \Phi(\mathrm{tp}_{\mathrm{qf}}(\bar{t}, \varnothing, \mathbf{J}))$.

Proof. Let $I = (Q^{\mathbf{J}}, <_{\mathbf{J}})$. By assumption (cf. Definition 3.3) there is a sequence $\langle \varphi'_n(x, \bar{y}_n) : n < \omega \rangle$. Let $k_n = \mathrm{lh}(\bar{y}_n)$.

Let I be an infinite linear order. Easily we can find $M_1 \models T_1$ and a sequence $\langle \bar{a}_t : t \in I \rangle$ with $\bar{a}_t \in {}^\omega(M_1)$ such that for every $\eta \in {}^\omega I$, the set $\{\varphi_n(\bar{x}, \bar{a}_t)^{\mathrm{if}(\eta(n)=t)} : t \in I, n < \omega\}$ is a type, i.e., finitely satisfiable in M_1.

Now by Ramsey's theorem, without loss of generality, $\langle \bar{a}_t : t \in I \rangle$ is an indiscernible sequence in M_1. Without loss of generality M_1 is λ^+-saturated, we then expand M_1 to M_1^+ by function $F_n^{M_1^+}$ $(n < \omega)$, (of finite arity) such that for $t_0 <_{\mathbf{J}} \ldots <_{\mathbf{J}} t_{n-1}$ from $Q^{\mathbf{J}}$ the element $F_n(\bar{a}_{t_0}, \bar{a}_{t_1}, \ldots \bar{a}_{t_{n-1}})$ or more exactly $F_n(\bar{a}_{t_0} {\restriction} k_0, \bar{a}_{t_1} {\restriction} k_1, \ldots, \bar{a}_{t_{n-1}} {\restriction} k_{n-1})$ realizes in M_1 the type $\{\varphi_\ell(x, \bar{a}_t)^{\mathrm{if}(\eta(\ell)=t)} : t \in I, \ell < n\}$. Let M_2^+ be an expansion of M_1^+ by Skolem functions such that $|\tau_{M_2^+}| = |T_1|$, (natural, though not strictly required). Without loss of generality $\langle \bar{a}_t : t \in I \rangle$ is an indiscernible sequence also in M_2^+.

[5] We may use finite $\bar{x}$ as usual. This does not matter by [7, 2.1].

Let D be a non-principal ultrafilter on ω and in $M_3^+ = (M_2^+)^\omega/D$, we let $\bar{a}'_t = \langle \bar{a}_t : n < \omega\rangle/D$ for $t \in I$, and $\bar{a}'_\eta = \langle F_n(\bar{a}_{\eta(0)}, \bar{a}_{\eta(1)}, \ldots, \bar{a}_{\eta(n-1)}) : n < \omega\rangle/D$ for $\eta \in \mathrm{incr}_\omega(I)$ and $\bar{a}'_t = \bar{a}'_{<F_n^{\mathbf{J}}(t) : n<\omega>}$ for $t \in P^{\mathbf{J}}$.

Let M_4^+ be the submodel of M_3^+ generated by $\{\bar{a}'_t : t \in \mathbf{J}\}$ and M be $M_4^+ \restriction \tau(T_1)$. Now $M, \langle \bar{a}_t : t \in \mathbf{J}\rangle$ are as required. Q.E.D.

Proposition 3.5. Assume $\mathbf{J}_\ell \in K_{\mathrm{oi}}$, and that M_ℓ, $\bar{\varphi}$, T_1, and T are as in Proposition 3.4 for $\ell = 1, 2$. Suppose that the following holds:

If f is a function from $\mathbf{J}_1$ (i.e., its universe) into $\mathcal{M}_{|T_1|,\aleph_0}(\mathbf{J}_2)$ (i.e., the free algebra generated by $\{x_t : t \in \mathbf{J}_2\}$ in the vocabulary $\tau_{|T_1|,\aleph_0} = \{F^n_\alpha : n < \omega \text{ and } \alpha < |T_1|\}, F^n_\alpha$ has arity n, cf. [15, III,§1] = [16]), we can find $t \in P^{\mathbf{J}_1}, n < \omega$, and $s_1, s_2 \in Q^{\mathbf{J}_1}$ and $k, \sigma, r^\ell_i (\ell = 1, 2 \text{ and } i < k), m, \sigma^*$ such that:

- $F_n^{\mathbf{J}_1}(t) = s_1 \neq s_2$,
- for $\ell \in \{1, 2\}$ we have $f(s_\ell) = \sigma(r^\ell_0, \ldots, r^\ell_{k-1})$ so $k < \omega, r^\ell_t \in \mathbf{J}_2$ for $i < k$ and σ is a $\tau_{|T_1|,\aleph_0}$-term not dependent on ℓ,
- $f(t) = \sigma^*(r_0, \ldots, r_{m-1}), \sigma^*$ is a $\tau_{|T_1|,\aleph_0}$-term and $r_0, \ldots, r_{m-1} \in \mathbf{J}_2$,
- the sequences $\langle r^1_i : i < k\rangle^\frown\langle r_i : i < m\rangle$ and $\langle r^2_i : i < k\rangle^\frown\langle r_i : i < m\rangle$ realize the same quantifier free type in $\mathbf{J}_2$.[6]

Then $M_1 \restriction \tau_T \ncong M_2 \restriction \tau_T$.

Proof. Straightforward (or as in [15, Ch.III] = [16]). Q.E.D.

We could have replaced $Q^{\mathbf{J}}$ by the disjoint union of $\langle Q^{\mathbf{J}}_n : n < \omega\rangle$, where $<^{\mathbf{J}}$ linearly orders each $Q^{\mathbf{J}}_n$ and $<^{\mathbf{J}} = \bigcup\{< \restriction Q^{\mathbf{J}_1}_n : n < \omega\}$. In this case, use Q_n to index parameters for $\varphi_n(x, \bar{y}_n)$). This plays no role in the present paper. For our present purpose, we can replace "not strongly stable" by a weaker demand. We present this briefly without detail. Recall (from [7, §5]) the following definition:

Definition 3.6. A (complete first order) theory T is *not strongly*$_4$ *dependent* if there is a sequence $\bar{\varphi} = \langle \varphi_n(\bar{x}, \bar{y}_n) : n < \omega\rangle$, (finite $\bar{x}$ of length $m < \omega$, as usual) of (first order) formulas from $\mathcal{L}(\tau_T)$, an infinite linear order I, a sequence $\langle \bar{a}_\eta : \eta \in \mathrm{incr}_{<\omega}(I)\rangle$ indiscernible in M with $\mathrm{lh}(\bar{a}_\eta) \leq \omega$ and letting $B = \bigcup\{\bar{a}_\eta : \eta \in \mathrm{incr}_{<\omega}(I)\}$ for some $m < \omega$ and $p \in \mathbf{S}^m(B, M)$

[6] We should close by the $F^{\mathbf{J}_2}_n$'s, but no need to iterate as $F^{\mathbf{J}_2}_n \restriction Q^{\mathbf{J}_2}$ is the identity so quantifier free type mean the truth value of the inequalities $F_{n_1}(r') \neq F_{n_2}(r')$ (including F_ω) and the order between those terms.

for every $k < \omega$ there is $n < \omega$, satisfying: for no linear order I^+ extending I and subset I_0 of I^+ with $\leq k$ members, do we have:

$$\text{if } \bar{t}^1, \bar{t}^2 \text{ are increasing sequences from } I \text{ of the same length } n \text{ realizing the same quantifier free type over } I_0 \text{ in } I^+ \text{ and for } i = 1,2 \text{ we let } \bar{b}^i = (\ldots {}^\frown(\bar{a}_{\langle t^i_{\eta(\ell)} \,:\, \ell < \mathrm{lh}(\eta)\rangle} \restriction n)^\frown \ldots)_{\eta \in \mathrm{incr}_{<\omega}(n)} \text{ then } \ell < n \wedge u \subseteq \mathrm{lh}(\bar{b}^1) \wedge |u| = \mathrm{lh}(\bar{y}_\ell) \text{ implies } \varphi_\ell(x, \bar{b}^1 \restriction u) \in p \Leftrightarrow \varphi_\ell(\bar{x}, \bar{b}^2 \restriction u) \in p. \tag{$\otimes$}$$

In the above, without loss of generality, $\bar{y}_n \triangleleft \bar{y}_{n+1}$ for $n < \omega$. A theory T is *strongly$_4$ stable* if it is stable and strongly dependent.

We can write the condition in Definition 3.6 without I^+ speaking about finite sets as done in $(*)$ in the proof of Proposition 3.7 below. We furthermore can get such $\langle \bar{a}'_\rho : \rho \in \mathrm{incr}_{<\omega}(I') \rangle$ for any infinite linear order I' by compactness.

Next we deduce a consequence of being non-strongly$_4$-dependent helpful in proving non-structure results.

Proposition 3.7. If $T_1 \supseteq T$ are complete first order theories, without loss of generality with Skolem functions and T is not strongly$_4$ dependent as witnessed by $\bar{\varphi} = \langle \varphi_n(\bar{x}, \bar{y}_n) : n < \omega \rangle$, i.e., as in Definition 3.6, then there is $\tau_1 \supseteq \tau_{T_1}, |\tau_1| = |T_1|$ and $\bar{\sigma}_n(\bar{z}_n) = \langle \sigma_{n,\ell}(\bar{z}_n) : \ell < \mathrm{lh}(\bar{\sigma}_n) \rangle, \sigma_{n,\ell}$ is a τ_1-term such that: if $I, \mathfrak{S}$ and $\mathbf{J} = \mathbf{J}_{I,\mathfrak{S}}$ are as in Definition 3.1(2), then there are M_1 and $\langle \bar{a}_t : t \in I \rangle$ and $\langle \bar{a}_\eta : \eta \in \mathfrak{S} \rangle$ such that:

- (α) M_1 is a τ_1-model and is the Skolem hull of $\{\bar{a}_t : t \in I\} \cup \{\bar{a}_\eta : \eta \in \mathfrak{S}\}$ (we write $\bar{a}_t$ for $t \in \mathfrak{S} \subseteq \mathbf{J}$ for uniformity),
- (β) $\langle \bar{a}_t : t \in \mathbf{J} \rangle$ is indiscernible in M_1,
- (γ) if $\eta \in \mathfrak{S}$ and $k < \omega$ then for large enough $n(*)$, if $u \subseteq n(*), |u| \leq k$, then we can find $\bar{s}, \bar{t}$ and $n_* < n(*)$ and $\bar{\sigma}$ such that
  - $\bar{s}, \bar{t}$ are sequences of members of $\{F^{\mathbf{J}}_n(\eta) : n < n(*)\}$,
  - $\mathrm{lh}(\bar{s}) = \mathrm{lh}(\bar{t}) \leq n(*)$,
  - $s_i <_I s_j \Leftrightarrow t_i <_I t_j$ for $i, j < \mathrm{lh}(\bar{s})$,
  - if $i < \mathrm{lh}(\bar{s}) = \mathrm{lh}(\bar{t})$ then $(\forall n \in u)(F^{\mathbf{J}}_n(\eta) \leq_I s_i \equiv F^{\mathbf{J}}_n(\eta) \leq_I t_i)$,
  - $\bar{\sigma} = \langle \sigma_i(\bar{y}) : i < \mathrm{lh}(\bar{y}_{n_*}) \rangle, \sigma_i$ a τ_1-term,
  - $M_1 \models \varphi_{n_*}[\bar{a}_\eta, \ldots, \sigma^{M_1}_i(\bar{a}_{t_0}, \bar{a}_{t_1}, \ldots), \ldots]_{i < \mathrm{lh}(\bar{y}_{n_*})} \equiv \neg\varphi_{n_*}[\bar{a}_\eta, \ldots, \sigma^{M_1}_i(\bar{a}_{s_0}, \bar{a}_{s_1}, \ldots), \ldots]_{i < \mathrm{lh}(\bar{y}_{n_*})}$,
- (δ) M_1 is a model of T_1 so $\tau_{M_1} \supseteq \tau_{T_1}$.

Proof. Fix $I, \mathfrak{S}$; without loss of generality, I is dense with neither first nor last element and is $\aleph_1$-homogeneous hence there are infinite increasing sequences of members of I.

Let $I, \langle \varphi_n(\bar{x}, \bar{y}_n) : n < \omega \rangle, \langle \bar{a}_\eta : \eta \in \mathrm{incr}_{<\omega}(I) \rangle$ and $p \in \mathbf{S}^m(\bigcup\{\bar{a}_\eta : \eta \in \mathrm{incr}_{<\omega}(I)\})$ exemplify T is not strongly$_4$ dependent, i.e., be as in the Definition so $m = \mathrm{lh}(\bar{x})$. For notational simplicity (and even without loss of generality, by [7, §5]) assume $m = 1$.

Now in Definition 3.6 we can add (by compactness) that

> there is a sequence $\langle (n_k, m_k, I^*_k) : k < \omega \rangle$ such that $k < n_k < m_k, m_k < m_{k+1}, I^*_k \subseteq I$ has m_k members, for no $I_0 \subseteq I^*_k$ with $\leq k$ members does $(\otimes)$ from Definition 3.6 hold for $\bar{t}^1, \bar{t}^2 \in \mathrm{incr}_{<n_k}(I^*_k)$ and (k, n_k) here standing for k, n. $(*)$

Without loss of generality, I is the reduct to the vocabulary $\{<\}$, i.e., to just a linear order of an ordered field $\mathbb{F}$ and $t_q \in \mathbb{F}$ for $q \in \mathbb{Q}$ are such that $0 <_{\mathbb{F}} t_q, (t_{q_1})^2 <_{\mathbb{F}} t_{q_2}$ for $q_1 <_{\mathbb{F}} q_2$ (hence $n < \omega \Rightarrow n <_{\mathbb{F}} t^n_{q_1} <_{\mathbb{F}} t_{q_2}$). By easy manipulation, without loss of generality, $I^*_k = \{t_i : i = 0, 1, \ldots, m_k\}$.

Now for each $m < \omega$ and $\eta \in \mathrm{incr}_m(I)$ we can choose c_η such that if $m = m_k$ then for some automorphism h of I mapping I^*_k onto $\mathrm{Rang}(\eta)$, letting $\hat{h}$ be an automorphism of M_1 mapping $\bar{a}_\nu$ to $\bar{a}_{h(\nu)}$ for $\nu \in \mathrm{incr}_{<\omega}(I)$, the element c_η realizes $\hat{h}(p)$ and $\langle c_\eta : \eta \in \mathrm{incr}_{<\omega}(I) \rangle$ is without repetitions.

Now, without loss of generality, $\big\langle \langle c_\eta \rangle^\frown \bar{a}_\eta : \eta \in \mathrm{incr}_{<\omega}(I) \big\rangle$ is an indiscernible sequence and let $a_t = c_{<t>}$ be such that M_0 be a model of T_1 satisfying $\bigcup\{\langle c_\eta \rangle^\frown \bar{a}_\eta : \eta \in \mathrm{incr}_{<\omega}(I)\} \subseteq M_0 {\restriction} \tau \prec \mathfrak{C}$. Without loss of generality $\langle \langle c_\eta \rangle\hat{\ } \bar{a}_\eta : \eta \in \mathrm{incr}_{<\omega}(I) \rangle$ is indiscernible in M_0 and we can find an expansion M_1 of M_0 such that $|\tau_{M_1}| = |T_1|$,

$$\bar{a}_\eta = \langle F_{\mathrm{lh}(\eta), i}(\bar{a}_{\eta(0)}, \ldots, \bar{a}_{\eta(n-1)}) : i < \mathrm{lh}(\bar{a}_\eta) \rangle, \text{ and}$$

$c_\eta = F_{\mathrm{lh}(\eta)}(\bar{a}_{\eta(0)}, \ldots, \bar{a}_{\eta(n-1)})$ if $\eta \in \mathrm{incr}_n(I)$, and M_1 has Skolem functions.

By manipulating I, without loss of generality, we can find $I_* \subseteq I$ of order type ω. So for some $H_n \in \tau_1$ for $n < \omega$, if $t_0 < t_1 < \ldots$ list I_*, for every $k < \omega$ large enough, for every $u \subseteq n(*)$ satisfying $|u| \leq k$ for every n large enough $H^{M_1}_n(\bar{a}_{t_0}, \bar{a}_{t_1}, \ldots, \bar{a}_{t_{n-1}})$ satisfies the demand (on the singleton $\bar{a}_\eta$ from clause (γ) in the claim).

Let D be a non-principal ultrafilter on ω such that $\{m_k : k < \omega\} \in D$, let M_2 be isomorphic to M^ω_1/D over M_1, i.e., $M_1 \prec M_2$ and there is an isomorphism f from M_2 onto M^ω_1/D extending the canonical embedding. If η is an increasing ω-sequence of members of I, we let

$$a^n_\eta = H^{M_1}_n(\bar{a}_{\eta(0)}, \ldots, \bar{a}_{\eta(n-1)}) \in M_1$$

and let

$$a_\eta = f^{-1}(\langle a^0_\eta, a^1_\eta, \ldots, a^n_\eta, \ldots : n < \omega \rangle / D) \in M_2.$$

Let M_2' be the Skolem hull of $\{\bar{a}_t : t \in I\} \cup \{a_\eta : \eta \in \mathfrak{S}\}$ inside M_2. It is easy to check that it is as required. Q.E.D.

It is helpful to have a sufficient condition for the non-isomorphism of two such models:

Proposition 3.8. Assume $\mathbf{J}_\ell \in K^{\rm oi}$, and $M_\ell, \bar{\varphi}, T_1, T$ as in Proposition 3.7 for $\ell = 1, 2$. Suppose that the following holds: if f is a function from $\mathbf{J}_1$ (i.e., its universe) into $\mathcal{M}_{|T_1|,\aleph_0}(\mathbf{J}_2)$ (i.e., the free algebra generated by $\{x_t : t \in \mathbf{J}_1\}$ the vocabulary $\tau_{|T_1|,\aleph_0} = \{F^n_\alpha : n < \omega \text{ and } \alpha < |T_1|\}, F^n_\alpha$ has arity n, we can find $t \in P^{\mathbf{J}_1}$ and $k_* < \omega$ such that for every $n_* < \omega$ we can find $\bar{s}_1, \bar{s}_2$ such that

- $\bar{s}_1, \bar{s}_2 \in {}^k I$ are increasing, $\bar{s}_1 = \langle F^{\mathbf{J}}_n(t) : n < n_* \rangle$ and $n < k_* \Rightarrow s_{2,n} = s_{1,n}$ and $s_{1,n_*-1} <_I s_{2,k_*}$,
- $f(\bar{s}_\ell) = \bar{\sigma}(r^\ell_0, \ldots, r^\ell_{k-1})$ so $k < \omega, r^\ell_t \in \mathbf{J}_2$ for $i < k$ so σ is a $\tau_{|T_1|,\aleph_0}$-term not dependent on ℓ,
- $f(t) = \sigma^*(r_0, \ldots, r_{m-1}), \sigma^*$ is a $\tau_{|T_1|,\aleph_0}$-term and $r_0, \ldots, r_{m-1} \in \mathbf{J}_2$,
- the sequences $\langle r^1_i : i < k \rangle^\frown \langle r_i : i < m \rangle$ and $\langle r^2_i : i < k \rangle^\frown \langle r_i : i < m \rangle$ realize the same quantifier free type in $\mathbf{J}_2$ (note: we should close by the $F^{\mathbf{J}_2}_n$, so type mean the truth value of the inequalities $F_{n_1}(r') \neq F_{n_2}(r')$ (including F_ω) and the order between those terms).

Then $M_1 \not\cong M_2$.

Proof. As in [10, III] or better in [15, III] = [16], called unembeddability. Q.E.D.

4 Forcing EF$^+$-equivalent Consistency non-isomorphic models

The following result is not optimal, but it is enough to prove necessary conditions on T for being lean and even on $(T, *)$. As for unstable T, cf. below in § 5.

Theorem 4.1. Assume $(\bar{\varphi}, T, T_1, \Phi)$ is as in Proposition 3.4, T stable and $\lambda = \lambda^{<\lambda} \geq \aleph_1 + |T_1|$ and $\mu = \lambda^+ > \lambda$. Then for some λ-complete λ^+-c.c. forcing notion $\mathbb{Q}$ we have that $\Vdash_{\mathbb{Q}}$ "there are models M_1, M_2 of T of cardinality λ^+ such that $M_1 {\restriction} \tau(T), M_2 {\restriction} \tau(T)$ are $\mathrm{EF}^+_{\alpha,\lambda,\lambda^+}$-equivalent for every $\alpha < \lambda$ but are not isomorphic".

We expect that we can improve Theorem 4.1 by allowing $\alpha < \lambda^+$ and replacing forcing by assuming, e.g., $2^\lambda = \lambda^+$,$\lambda = \lambda^{<\lambda}$. We shall continue this line of research in [12].

Proof. We define a partial order $\mathbb{Q}$. A condition p is in $\mathbb{Q}$ if it consists of the following objects satisfying the following conditions:

(a) $u = u^p \in [\mu]^{<\lambda}$ such that if $\alpha + i \in u$ and $i < \lambda$, then $\alpha \in u$;

(b) $<^p$ a linear order of u such that if $\alpha, \beta \in u$ and $\alpha + \lambda \leq \beta$, then $\alpha <^p \beta$;

(c) for $\ell = 1, 2$, $\mathfrak{S}^p_\ell$ is a subset of $\{\eta \in {}^\omega u : \eta(n) + \lambda \leq \eta(n+1) \text{ for } n < \omega\}$ such that $\eta \neq \nu \in \mathfrak{S}^p_\ell \Rightarrow \mathrm{Rang}(\eta) \cap \mathrm{Rang}(\nu)$ is finite; note that in particular $\eta \in \mathfrak{S}^p_\ell$ is without repetitions and is $<^p$-increasing;

(d) Λ^p a set of $< \lambda$ increasing sequences of ordinals from $\{\alpha \in u^p : \lambda | \alpha\}$ hence of length $< \lambda$;

(e) $\bar{f}^p = \langle f^p_\rho : \rho \in \Lambda^p \rangle$ such that

1. f^p_ρ is a partial automorphism of the linear order $(u^p, <^p)$ such that $\alpha \in \mathrm{Dom}(f^p_\rho) \Rightarrow \alpha + \lambda = f^p_\rho(\alpha) + \lambda$ and we let $f^{1,p}_\rho = f^p_\rho, f^{2,p}_\rho = (f^p_\rho)^{-1}$;
2. if $\eta \in \mathfrak{S}^p_\ell, \rho \in \Lambda^p, \ell \in \{1, 2\}$ then $\mathrm{Rang}(\eta)$ is included in $\mathrm{Dom}(f^{\ell,p}_\rho)$ or is almost disjoint to it (i.e., except finitely many "errors");
3. if $\rho \triangleleft \varrho \in \Lambda^p$ then $\rho \in \Lambda^p$ and $f^p_\rho \subseteq f^p_\varrho$
4. $f^{\ell,p}_\varnothing$ is the empty function and if $\rho \in \Lambda^p$ has limit length then $f^p_\rho = \cup\{f^p_{\rho \restriction i} : i < \mathrm{lh}(\rho)\}$.
5. if $\rho \in \Lambda^p$ has length $i + 1$ then $\mathrm{Dom}(f^{\ell,p}_\rho) \subseteq \rho(i)$ for $\ell = 1, 2$;
6. if $\rho \in \Lambda^p$ and $\eta \in {}^\omega(\mathrm{Dom}(f^p_\rho))$ then $\eta \in \mathfrak{S}^p_1 \Leftrightarrow \langle f^p_\rho(\eta(n)) : n < \omega\rangle \in \mathfrak{S}^p_2$;
7. if $\rho_n \in \Lambda^p$ for $n < \omega$ and $\rho_n \triangleleft \rho_{n+1}$ and $\lambda > \aleph_0$ then $\bigcup\{\rho_n : n < \omega\} \in \Lambda$.

We now define the order $\leq = \leq_{\mathbb{Q}}$ on $\mathbb{Q}$. We fix $p, q \in \mathbb{Q}$ and let $p \leq q$ if $u^p \subseteq u^q$, $\leq^p = \leq^q \restriction u^p$, $\mathfrak{S}^p_\ell \subseteq \mathfrak{S}^q_\ell$ (for $\ell = 1, 2$), $\Lambda^p \subseteq \Lambda^q$, and the following conditions hold:

(a) if $\rho \in \Lambda^p$ then $f^p_\rho \subseteq f^q_\rho$;

(b) if $\eta \in \mathfrak{S}^q_\ell \setminus \mathfrak{S}^p_\ell$ then $\mathrm{Rang}(\eta) \cap u^p$ is finite;

(c) if $\rho \in \Lambda^p$ and $f^p_\rho \neq f^q_\rho$ then $u^p \cap \sup \mathrm{Rang}(\rho) \subseteq \mathrm{Dom}(f^{\ell,q}_\rho)$ for $\ell = 1, 2$; and

(d) if $\rho \in \Lambda^p$ and $\ell \in \{1, 2\}, \alpha \in u^p \setminus \mathrm{Dom}(f^{\ell,p}_\rho)$ and $\alpha \in \mathrm{Dom}(f^{\ell,q}_\rho)$ then $f^{\ell,p}_\rho(\alpha) \notin u^p$.

Having defined the forcing notion $\mathbb{Q}$, we can now start to investigate it. We obviously have that $\mathbb{Q}$ is a partial order of cardinality $\mu^{<\lambda} = \lambda^+$.

Claim 4.2. If $\bar{p} = \langle p_i : i < \delta \rangle$ is $\leq^{\mathbb{Q}}$-increasing, δ a limit ordinal $< \lambda$ of uncountable cofinality then $p_\delta := \bigcup\{p_i : i < \delta\}$ defined naturally is an upper bound of $\bar{p}$.

Proof. Think about it, or consider the proof of Claim 4.3. The case $\mathrm{cf}(\delta) > \aleph_0$ is easier because of clauses (e4) and (e7). Q.E.D.

Claim 4.3. If $\delta < \lambda$ is a limit ordinal of cofinality $\aleph_0$ and the sequence $\bar{p} = \langle p_i : i < \delta \rangle$ is increasing (in $\mathbb{Q}$), then it has an upper bound.

Proof. We define $p_\delta \in \mathbb{Q}$ as follows: $u^{p_\delta} = \bigcup\{u^{p_i} : i < \delta\}$, $<^{p_\delta} = \bigcup\{<^{p_i} : i < \delta\}, \Lambda^{p_\delta} = \bigcup\{\Lambda^{p_i} : i < \delta\} \cup \{\rho : \rho$ is an increasing sequence of ordinals from u^{p_δ} of length a limit ordinal of cofinality $\aleph_0$ such that $\rho{\restriction}\varepsilon \in \cup\{\Lambda^{p_i} : i < \delta\}\}$ for all $\varepsilon < \mathrm{lh}(\rho)$.

Let $\bar{f}^{p_\delta} = \langle f^{p_\delta}_\rho : \rho \in \Lambda^q \rangle$ be such that if $i < \delta$ and $\rho \in \Lambda^{p_i} \setminus \bigcup\{\Lambda^{p_j} : j < i\}$, then $f^q_\rho = \bigcup\{f^{p_j}_\rho : j \in [i, \delta)\}$ and if $\rho \in \Lambda^{p_\delta} \setminus \{\Lambda^{p_i} : i < \delta\}$ then $f^{p_\delta}_\rho = \bigcup\{f^{p_\delta}_{\rho\restriction\varepsilon} : \varepsilon < \mathrm{lh}(\rho)\}$ is well defined as $\varepsilon < \mathrm{lh}(\rho) \Rightarrow \rho^\frown\langle\varepsilon\rangle \in \bigcup\{\Lambda^{p_j}; j < \delta\}$. Clearly clauses (a), (b), (d), (e1), (e3), (e4), (e5), and (e7) for $p_\delta \in \mathbb{Q}$ hold. Lastly, let $\mathfrak{S}^{p_\delta}_\ell = \bigcup\{\mathfrak{S}^{p_\alpha}_\ell : \alpha < \delta\}$ for $\ell = 1, 2$.

Subclaim 4.4. If $\rho \in \Lambda^{p_\delta} \setminus \bigcup\{\Lambda^{p_\alpha} : \alpha < \delta\}$ then $\mathrm{Dom}(f^{p_\delta}_\rho) = u^{p_\delta} \cap \sup \mathrm{Rang}(\rho) = \mathrm{Rang}(f^{p_\delta}_\rho)$ and for every $\alpha < \delta$ for some $\beta < \delta$ we have $f^{p_\delta}_\rho{\restriction}u^{p_\alpha} \subseteq f^{p_\beta}_{\rho\restriction i}$ for some $i < \mathrm{lh}(\rho)$.

Proof. Clearly, $\mathrm{cf}(\mathrm{lh}(\rho)) = \aleph_0$. Assume $\alpha < \delta$ and $i < \mathrm{lh}(\rho)$. Clearly for some $\beta \in (\alpha, \delta)$ we have $\rho{\restriction}i \in \Lambda^{p_\beta}$. Also the set $\{j < \mathrm{lh}(\rho) : \rho{\restriction}j \in \Lambda^{p_\beta}\}$ is an initial segment of $\mathrm{lh}(\rho)$ and cannot be $\mathrm{lh}(\rho)$ because $\rho \notin \Lambda^{p_\beta}$ by clause (e7) of the definition of $\mathbb{Q}$. So for some $j < \mathrm{lh}(\rho)$ we have $\rho{\restriction}j \notin \Lambda^{p_\beta}$ but by the choice of ρ for some $\gamma < \delta$ we have $\rho{\restriction}j \in \Lambda^{p_\gamma}$, so necessarily $\beta < \gamma$. As $p_\alpha \leq_{\mathbb{Q}} p_\beta \leq_{\mathbb{Q}} p_\gamma$ by clause (c) of the definition of $\leq_{\mathbb{Q}}$, as $\rho{\restriction}i \in \Lambda^{p_\gamma} \setminus \Lambda^{p_\beta}$ we know that $u^{p_\beta} \cap \sup \mathrm{Rang}(\rho{\restriction}i)$ is included in $\mathrm{Dom}(f^{\ell,p_\gamma}_{\rho\restriction i})$ for $\ell = 1, 2$ by $p_\alpha \leq_{\mathbb{Q}} p_\beta$ hence $u^{p_\alpha} \cap \sup \mathrm{Rang}(\rho{\restriction}i)$ is included in $\mathrm{Dom}(f^{\ell,p_\gamma}_{\rho\restriction i_\alpha})$ which $\subseteq \mathrm{Dom}(f^{\ell,p_\delta}_{\rho\restriction i})$ for $\ell = 1, 2$.

As this holds for any $\alpha < \delta$ and $i < \mathrm{lh}(\rho)$ and $u^{p_\delta} \cap \sup \mathrm{Rang}(\rho{\restriction}i) = \bigcup\{u^{p_\alpha} \cap \sup \mathrm{Rang}(\rho) : \alpha < \delta\}$ it follows that for $\ell = 1, 2$ we have that $\varepsilon \in u^{p_\delta} \cap \sup \mathrm{Rang}(\rho)$ implies $(\exists \alpha < \delta)(\varepsilon \in u^{p_\alpha} \cap \sup \mathrm{Rang}(\rho))$. That implies $(\exists \beta < \delta)[\varepsilon \in \mathrm{Dom}(f^{\ell,p_\delta}_\rho)]$, and this finally implies $\varepsilon \in \mathrm{Dom}(f^{\ell,p_\delta}_\rho)$, so are done. Q.E.D.

Subclaim 4.5. If $\rho \in \bigcup\{\Lambda^{p_\alpha} : \alpha < \delta\}$ then exactly one of the following occurs:

(a) There is a unique $\alpha = \alpha(\rho) < \delta$ such that $\rho \in \Lambda^{p_\alpha}$, $(\forall \beta)(\alpha \leq \beta < \delta \Rightarrow f_\rho^{p_\beta} = f_\rho^{p_\alpha})$ and $(\forall \beta < \alpha)(\rho \in \Lambda^{p_\beta} \rightarrow f_\rho^{p_\beta} \neq f_\rho^{p_\alpha})$.

(b) $\mathrm{Dom}(f_\rho^{p_\delta}) = u^{p_\delta} \cap \sup \mathrm{Rang}(\rho) = \mathrm{Rang}(f_\rho^{p_\delta})$ and $(\forall \alpha < \delta)(\exists \beta < \delta)(f_\rho^{p_\delta} \restriction u^{p_\alpha} \subseteq f_\rho^{p_\beta})$.

Proof. Similar to the proof of Subclaim 4.4. Q.E.D.

In order to finish the proof of $p_\delta \in \mathbb{Q}$, it remains to check clauses (c), (e2), and (e6).

Clause (c). Obvious by the choice of $\mathfrak{S}_1^{p_\delta}$.

Clause (e2). So let $\eta \in \mathfrak{S}_\ell^{p_\delta}, \rho \in \Lambda^{p_\delta}$ where $\ell \in \{1, 2\}$ and we should prove that $\mathrm{Rang}(\eta) \subseteq \mathrm{Dom}(f_\rho^{\ell,p,\delta})$ or $\mathrm{Rang}(\eta) \cap \mathrm{Dom}(f_\rho^{\ell,p})$ is finite. For some $\alpha < \delta$ we have $\eta \in \mathfrak{S}_\ell^{p_\alpha}$. If $\rho \in \bigcup \{\Lambda^{p_\beta} : \beta < \delta\}$ then we apply Subclaim 4.5, now if clause (a) there holds so $\alpha = \alpha(\rho) < \delta$ is well defined and we use $p_\alpha \in \mathbb{Q}$ and if clause (b) there holds then trivially $\mathrm{Rang}(\eta) \subseteq u^{p_\delta} \subseteq \mathrm{Dom}(f_\rho^{\ell,p_\delta})$ so assume $\rho \in \Lambda^{p_\delta} \setminus \bigcup \{\Lambda^{p_\beta} : \beta < \delta\}$.

By Subclaim 4.4 we finish as in the case if (b) from Subclaim 4.5 holds.

Clause (e6). By the choice of $\mathfrak{S}^{p_\delta}$ and the proof of clause (e2).

We now have to check that for $\alpha < \delta$, the pair (p_α, p_δ) satisfies the definition of $\leq_{\mathbb{Q}}$ which is straightforward. Q.E.D. (Claim 4.3)

Claim 4.6. If $\alpha < \mu$ then $\mathcal{I}_\alpha^1 := \{p \in \mathbb{Q} : \alpha \in u^p\}$ is dense and open as well as $\mathcal{I}_* = \{p \in \mathbb{Q}$: if $\delta \in u^p, \lambda | \delta$ and $\mathrm{cf}(\delta) < \lambda$ then $\delta = \sup(\delta \cap u)\}$.

Proof. The proof is straightforward. For the claim about $\mathcal{I}_\alpha^1$, given $p \in \mathbb{Q}$ we define $q \in \mathbb{Q}$ by $u^q := u^p \bigcup \{\beta \leq \alpha : \beta + \lambda = \alpha + \lambda\}$, $\alpha_1 < \alpha_2$ iff $\alpha_1 <^p \alpha_2$ or $\alpha_1 < \alpha_2 \wedge \{\alpha_1, \alpha_2\} \nsubseteq u^p \wedge \{\alpha_1, \alpha_2\} \subseteq u^q$, $\mathfrak{S}_\ell^q = \mathfrak{S}_\ell^p$ for $\ell = 1, 2$, $\Lambda^q = \Lambda^p$, and $f_\rho^q = f_\rho^p$ for $\rho \in \Lambda^q$. Now check. For the second claim about $\mathcal{I}_*$ use the first part and Claim 4.3. Q.E.D.

Claim 4.7. If $\varrho \in \Lambda^* := \{\rho : \rho$ is an increasing sequence of ordinals $< \lambda^+$ divisible by λ of length $< \lambda\}$ then $\mathcal{I}_\varrho^2 = \{p \in \mathbb{Q} : \varrho \in \Lambda^p\}$ is dense open.

Proof. Let $p \in \mathbb{Q}$, by Claims 4.6 and 4.3 there is $q \geq p$ (from $\mathbb{Q}$) such that $\mathrm{Rang}(\varrho) \subseteq u^q$. If $\varrho \in \Lambda^q$ we are done, otherwise define q' as follows: $u^{q'} = u^q, <^{q'} = <^q, \mathfrak{S}_\ell^{q'} = \mathfrak{S}_\ell^q, \Lambda^{q'} = \Lambda^q \cup \{\varrho \restriction \varepsilon : \varepsilon \leq \mathrm{lh}(\varrho)\}$ and if $i \leq \mathrm{lh}(\varrho), \varrho \restriction i \notin \Lambda^q$ then we let $f_{\varrho \restriction i}^{q'} = \cup \{f_\rho^q : \rho \in \Lambda^q$ and $\rho \triangleleft \varrho \restriction i\}$. We should check all the clauses in the definition of $\mathbb{Q}$ and, e.g., clause (e6) holds because q satisfies clause (e7). Then we should check all the clauses of "$q \leq_{\mathbb{Q}} q'$". Q.E.D.

Claim 4.8. If ϱ is as in Claim 4.9 and $\alpha < \lambda^+$ and $\ell \in \{1, 2\}$ then

$$\mathcal{I}^3_{\varrho,\alpha,\ell} = \{p \in \mathbb{Q} : \alpha \in \mathrm{Dom}(f^{\ell,p}_\varrho) \text{ so } \varrho \in \Lambda^p, \alpha \in u^p\} \text{ is dense open.}$$

Proof. By Claims 4.6 and 4.7. Q.E.D.

Claim 4.9. If $p \in \mathbb{Q}$ and $\varrho \in \Lambda^p$ then for some q we have $p \leq_{\mathbb{Q}} q \wedge f^q_\varrho \neq f^p_\varrho \wedge \{\alpha + \lambda : \alpha \in u^q\} = \{\alpha + \lambda : \alpha \in u^p\}$.

Proof. For each $\delta \in u \cap \sup \mathrm{Rang}(\varrho)$ divisible by λ let $u_\delta = u \cap [\delta, \delta + \lambda)$. So $g_\delta := f^p_\rho \restriction u_\delta$ is a partial function from u_δ into u_δ and $f^p_\rho = \bigcup\{g_\delta : \delta$ as above$\}$. Now, for δ as above we can find f_δ such that f_δ is a one-to-one function, $g_\delta = f^p_\varrho \restriction u_\delta \subseteq f_\delta$, $\alpha \in \mathrm{Dom}(f_\delta)$ iff $\alpha \in u_\delta \vee f_\delta(\alpha) \in u_\delta$, $\mathrm{Dom}(f_\delta) \backslash u_\delta$ is an initial segment $[\alpha^1_\delta, \alpha^2_\delta)$ of $[\delta, \delta + \lambda) \backslash u_\delta$, $\mathrm{Rang}(f_\delta) \backslash u$ is an initial segment $[\alpha^2_\delta, \alpha^3_\delta)$ of $[\delta, \delta + \lambda) \backslash u \backslash \mathrm{Dom}(f_\delta)$, f_δ maps $[\alpha^1_\delta, \alpha^2_\delta)$ onto $u_\delta \backslash \mathrm{Rang}(f^p_\varrho \restriction u_\delta)$, and f_δ maps $u_\delta \backslash \mathrm{Dom}(f^p_\varrho \restriction u_1)$ onto $[\alpha^2_\delta, \alpha^3_\delta)$. Now we can find a linear order $<_1$ on $u_\delta \cup [\alpha^1_\delta, \alpha^3_\delta]$ such that f_δ is order preserving (as the class of linear orders has amalgamation).

Lastly, we define q by $u^q = u^p \cup \{[\alpha^1_\delta, \alpha^3_\delta) : \delta \text{ as above}\}$, $\alpha <^q \beta$ iff $(\exists \delta)(\alpha <_\delta \beta)$ or $\alpha + \lambda \leq \beta$, $\Lambda^q = \Lambda^p$, and $\mathfrak{S}^q_\ell = \mathfrak{S}^p_\ell \cup \{\langle f^{3-\ell}_\rho(\eta(n)) : n < \omega\rangle : \ell \in \{1, 2\}, \rho \in \Lambda^p \text{ and } \eta \in \mathfrak{S}^p_{3-\ell}\}$.

Now we have to check that $q \in \mathbb{Q}$. This is straightforward; e.g. for clause (c), assume $\eta \neq \nu \in \mathfrak{S}^q_\ell$ and we have to prove that $\mathrm{Rang}(\eta) \cap \mathrm{Rang}(\nu)$ is finite.

Now we have four cases: first $\eta, \nu \in \mathfrak{S}^p_\ell$, so use clause (c) for ℓ. Second, $\eta, \nu \in \mathfrak{S}^q_\ell \backslash \mathfrak{S}^p_\ell$, so η, ν are images by $f^{3-\ell,q}_\rho$ of members of $\mathfrak{S}^p_{3-\ell}$, as this function is one-to-one, this follows from $p, \mathfrak{S}^p_{3-\ell}$ satisfying clause (c). Third, $\eta \in \mathfrak{S}^p_\ell \wedge \nu \in \mathfrak{S}^q_\ell \backslash \mathfrak{S}^p_\ell$, then $\nu = \langle f^{3-\ell,q}_\rho(\nu'(n)) : n < \omega\rangle$ for some $\nu' \in \mathfrak{S}^p_{3-\ell}$ satisfying $\mathrm{Rang}(\nu') \nsubseteq \mathrm{Dom}(f^{3-\ell,p}_\rho)$, hence for some $n_*, n \in [n_*, \omega) \Rightarrow \nu'(n) \notin \mathrm{Dom}(f^{3-\ell,p}_\rho) \Rightarrow \nu(n) \notin u^p$ but $\mathrm{Rang}(\eta) \subseteq u^p$ so we are done. Fourth, $\eta \in \mathfrak{S}^q_\ell \backslash \mathfrak{S}^p_\ell \wedge \nu \in \mathfrak{S}^p_\ell$ the proof is dual. The proof of clause (e2) is similar.

Also we have to check that $p \leq_{\mathbb{Q}} q$. This is straightforward, clause (b) is proved as in the proof of (c) of the definition of $\mathbb{Q}$ above and clause (d) holds by our choice of the f_δ's. Now check that q is as required. Q.E.D.

Let $\mathbb{Q}^+ = \{p \in \mathbb{Q}$: if $\ell \in \{1, 2\}$ and $\rho \in \Lambda^p$ then $\mathrm{Dom}(f^{\ell,p}_\rho) = u^p \cap \sup \mathrm{Rang}(\rho)\}$.

Claim 4.10. $\mathbb{Q}^+$ is a dense subset of $\mathbb{Q}$, moreover $(\forall p \in \mathbb{Q})(\exists q \in \mathbb{Q}^+)(p \leq q \wedge \bigcup\{\alpha + \lambda : \alpha \in u^q\} = \bigcup\{\alpha + \lambda : \alpha \in u^p\}]$.

Proof. Let $p \in \mathbb{Q}, \kappa = |\Lambda|, \delta = \kappa \times \kappa$ and $\{\rho_i : i < i_* < \lambda\}$ list Λ^p each appearing unboundedly often. We choose p_i by induction on $i \leq \delta$ such that $p_i \in \mathbb{Q}$, $j < i$ implies $p_i \leq_{\mathbb{Q}} p_j$, $p_0 = p$, $\Lambda^{p_i} = \Lambda^p$, $f^{p_{i+1}}_{\rho_i} \neq f^{p_i}_{\rho_i}$, and

$\bigcup\{\alpha+\lambda : \alpha \in u^{p_i}\} = \bigcup\{\alpha+\lambda : \alpha \in u^p\}$. For $i = 0$, use $p_0 = p$; for i limit, use Claim 4.3; for $i = j+1$, use Claim 4.9. Now p_δ is as required. Q.E.D.

Claim 4.11. For $p \in \mathbb{Q}$ and $\delta < \lambda^+$ divisible by $\lambda, p{\restriction}\delta$ is naturally defined, belongs to $\mathbb{Q}$ and $u^p \subseteq \delta$ implies $p{\restriction}\delta = p$ and $p{\restriction}\delta \leq_{\mathbb{Q}} p$, where $q = p{\restriction}\delta$ be defined by $u^q = u^p \cap \delta$, $<^q = <^p {\restriction}\delta$, $\mathfrak{S}^q_\ell = \{\eta \in \mathfrak{S}^p_\ell :\ \mathrm{Rang}(\eta) \subseteq \delta\}$, $\Lambda^q = \{\rho \in \Lambda^p : \sup \mathrm{Rang}(\rho) \leq \delta\}$, and $\bar{f}^q = \langle f^q_\rho : \rho \in \Lambda^q\rangle$ where $f^q_\rho = f^p_\rho$.

Proof. Why? Check. Q.E.D.

Claim 4.12. If $\delta < \lambda^+$ is divisible by $\lambda, p \in \mathbb{Q}^+$ and $(p{\restriction}\delta) \leq_{\mathbb{Q}} q \in \mathbb{Q}^+$ but $u^q \subseteq \delta$ then p, q are compatible in $\mathbb{Q}$, moreover has a common upper bound $r = p + q$ such that $r{\restriction}\delta = q \wedge u^r = u^p \cup u^q$.

Proof. Note that if $\rho \in \Lambda^p \cap \Lambda^q$ then $\sup \mathrm{Rang}(\rho) \leq \delta$ by clauses (e4) and (e5) of the definition of $\mathbb{Q}$; also $\Lambda^p \cap \Lambda^q = \Lambda_{p{\restriction}\delta}$. We define r by $u^r = u^p \cup u^q$, for $\alpha, \beta \in u^r$ we have $\alpha <^2 \beta$ iff $\alpha + \lambda \leq \beta$ or $\alpha <^q \beta$ or $\alpha <^p \beta$, $\mathfrak{S}^r_\ell$ is $\mathfrak{S}^p_\ell \cup \mathfrak{S}^q_\ell$ for $\ell = 1, 2$, $\Lambda^r = \Lambda^p \cup \Lambda^q$, and $\bar{f}^r = \langle f^r_\rho : \rho \in \Lambda^r\rangle$ where

$$f^r_\rho := \begin{cases} f^q_\rho & \text{if } \rho \in \Lambda^q \\ f^p_\rho \cup \bigcup\{f^q_{\rho{\restriction}i} : i \leq \mathrm{lh}(\rho) \text{ and } \rho{\restriction}i \in \Lambda^q\} & \text{if } \rho \in \Lambda^p \backslash \Lambda^q. \end{cases}$$

Why is $r \in \mathbb{Q}$? We should check all the clauses in the definition which are easy. E.g., in clause (c), $\eta \neq \nu \in \mathfrak{S}^r_\ell \Rightarrow \aleph_0 > |\mathrm{Rang}(\eta) \cap \mathrm{Rang}(\nu)|$, the only new case is $\eta \in \mathfrak{S}^p_\ell \Leftrightarrow \nu \notin \mathfrak{S}^p_\ell$ so, without loss of generality, $\eta \in \mathfrak{S}^p_\ell \backslash \mathfrak{S}^q_\ell \wedge \nu \in \mathfrak{S}^q_\ell$, hence $\sup(\eta) > \delta$ hence $\mathrm{Rang}(\eta) \cap \delta$ is finite but $\mathrm{Rang}(\nu) \subseteq u^q \subseteq \delta$.

Also clauses (e2) and (e6) should be checked only when f^r_ρ is new so necessarily $\rho \in \Lambda^p$ so $f^r_\rho = f^p_\rho \cup \bigcup\{f^q_{\rho{\restriction}i} : \rho{\restriction}i \in \Lambda^q\}$, but recalling that any $\eta \in \mathfrak{S}^r_\ell$ is an increasing ω-sequence, clearly if $\sup \mathrm{Rang}(\eta) > \delta$ we use "p satisfies clauses (e2) and (e6)" and if $\sup \mathrm{Rang}(\eta) \leq \delta$ we use "q satisfies clauses (e2), (e6), and (e7)".

Why do $p \leq_{\mathbb{Q}} r$ and $p \leq_{\mathbb{Q}} r$ hold? We should check all the clauses in the definition of $\leq_{\mathbb{Q}}$ for both pairs. They are easy, e.g., clause (b) holds because if $\eta \in \mathfrak{S}^r_\ell \backslash \mathfrak{S}^q_\ell$ then $\eta \in \mathfrak{S}^p_\ell \backslash S^q_\ell$ hence $\sup \mathrm{Rang}(\eta) > \delta$ and it should be clear; if $\eta \in \mathfrak{S}^r_\ell \backslash \mathfrak{S}^p_\ell$ then $\eta \in \mathfrak{S}^q_\ell \backslash \mathfrak{S}^p_\ell$ and we can use $p{\restriction}\delta \leq_{\mathbb{Q}} q$, i.e., clause (b) for this pair.

Concerning clause (c) for $p \leq_{\mathbb{Q}} r$, recall that $p, q \in \mathbb{Q}^+$ so if $\ell \in \{1, 2\}$ and $\rho \in \Lambda^p$ then $u^p = \mathrm{Dom}(f^{\ell,p}_\rho) \subseteq \mathrm{Dom}(f^{\ell,r}_\rho)$, so clause (c) is satisfied, and similarly clause (c) for $q \leq_{\mathbb{Q}} r$. Q.E.D.

Claim 4.13. $\mathbb{Q}$ satisfies the λ^+-c.c.

Proof. Let $p_\alpha \in \mathbb{Q}$ for $\alpha < \lambda^+$, so by Claim 4.10 there are q_α such that $p_\alpha \leq_{\mathbb{Q}} q_\alpha \in \mathbb{Q}^+$, now use the Δ-system lemma: $S^{\lambda^+}_\lambda = \{\delta < \lambda^+ :\ \mathrm{cf}(\delta) = \lambda\}$;

now if $\delta \in S^{\lambda^+}_{\lambda}$ then $p{\restriction}\delta \in \mathbb{Q} \wedge \sup(u^{p{\restriction}\delta}) < \delta$ and $\lambda \geq |\{p \in \mathbb{Q} : u^p = u\}$ for any u. Hence for some stationary $S \subseteq S^{\lambda^+}_{\lambda}$ and some p_* and for any $\delta \in S$ and $\delta_1 < \delta_2 \in S$, we get $q_\delta{\restriction}\delta = p_*$ and $\sup(u^{q_{\delta_2}}) < \delta_2$. So for any $\delta_2 < \delta_2$ from S by Claim 4.12 the condition $q_{\delta_1}, q_{\delta_2}$ are compatible. Q.E.D.

Define a $\mathbb{Q}$-name $\underset{\sim}{\mathbf{J}}_\ell \in K^{\rm oi}_{\mu}$, as follows: $Q^{\underset{\sim}{\mathbf{J}}_\ell} = \mu$, $\mathfrak{S}^{\underset{\sim}{\mathbf{J}}_\ell} = \cup\{\mathfrak{S}^p_\ell : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}}\}$, $<^{\underset{\sim}{\mathbf{J}}_\ell} = \cup\{<^p : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}}\}$, $F_n^{\underset{\sim}{\mathbf{J}}_\ell}$ is a unary function, the identity on λ^+ and $\eta \in \mathfrak{S}^{\underset{\sim}{\mathbf{J}}_\ell}$ implies $F_n^{\underset{\sim}{\mathbf{J}}_n}(\eta) = \eta(n)$. Now for $\ell \in \{1,2\}$ and $p \in \mathbb{Q}$ let $\mathbf{J}^p_\ell \in K_{\rm oi}$ be defined as follows: $\mathbf{J}^p_\ell$ has universe $u^p \cup \mathfrak{S}^p_\ell$, $<^{\mathbf{J}_\ell} = <^p$, $Q^{\mathbf{J}^p_\ell} = u^p$, and $F_n^{\mathbf{J}^p_\ell}(\eta) = \eta(n)$.

Claim 4.14.

(a) $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\mathbf{J}}_\ell \in K^{\rm oi}_{\lambda^+}$",

(b) $\Vdash_{\mathbb{Q}}$ "for each $\delta < \lambda^+$ divisible by λ the linear order $([\delta, \delta+\lambda), <^{\underset{\sim}{\mathbf{J}}_\ell} {\restriction}(\delta, \delta+\lambda))$ is a saturated linear order and $\alpha+\lambda \leq \beta < \lambda^+ \Rightarrow \alpha <^{\underset{\sim}{\mathbf{J}}_\ell} \beta$", and

(c) $p \in \mathbb{Q} \Rightarrow p \Vdash_{\mathbb{Q}}$ "$\mathbf{J}^p_\ell \subseteq \underset{\sim}{\mathbf{J}}_\ell$ for $\ell = 1, 2$".

Proof. Why? Think! Q.E.D.

Claim 4.15. If $\delta < \lambda^+$ is divisible by λ then $\Vdash$ "$\underset{\sim}{\mathbf{J}}_\ell{\restriction}\delta \in K^{\rm oi}_{\lambda}$" where $\mathbf{J}_\ell{\restriction}\delta = ((\delta \cup (P^{\mathbf{J}_\ell} \cap {}^{\omega}\delta), Q^{\mathbf{J}_\ell} \cap \delta, P^{\mathbf{J}_\ell}{\restriction}\delta, F_n^{\mathbf{J}_\ell}{\restriction}(\delta \cup (P^{\mathbf{J}_\ell} \cap {}^{\omega}\delta)))_{n<\omega}$".

Claim 4.16. $\Vdash_{\mathbb{Q}}$ "$\mathrm{EM}_{\tau(T)}(\underset{\sim}{\mathbf{J}}_1, \Phi), \mathrm{EM}_{\tau(T)}(\underset{\sim}{\mathbf{J}}_2, \Phi)$ are $\mathrm{EF}^+_{\lambda,\lambda^+}$-equivalent (so the games are of length $< \lambda$, and the player AIS chooses sets of cardinality $< \lambda^+$).

Proof. To show the $\mathrm{EF}^+_{\lambda,\lambda^+}$-equivalence, it suffices to show that $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\mathbf{J}}_1, \mathbf{J}_2$ are $\mathrm{EF}_{\lambda,\lambda^+}$-equivalent" by Proposition 2.9 as $\lambda \geq \aleph_1 + |T_1|$. From Claim 4.7, recall $\Lambda^* = \{\rho : \rho$ is an increasing sequence of ordinals $< \lambda^+$ divisible by λ of length $< \lambda\}$, (is the same in $\mathbf{V}$ and $\mathbf{V}^{\mathbb{Q}}$). For $\rho \in \Lambda^*$ let $\underset{\sim}{f}_\rho = \bigcup\{f^p_\rho : \rho \in \underset{\sim}{G}, p \in \Lambda^p\}$ and by clauses (e1) and (e5) in the definition of $\mathbb{Q}$ and (a) in the definition of $\leq_{\mathbb{Q}}$, we easily get $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{f}_\rho$ a partial isomorphism from $\underset{\sim}{\mathbf{J}}_1{\restriction}\sup \mathrm{Rang}(\rho)$ into $\underset{\sim}{\mathbf{J}}_2{\restriction}\sup \mathrm{Rang}(\rho)$" (cf. the definition in Claim 4.15).

Now $\Vdash_{\mathbb{Q}}$ "$\mathrm{Dom}(\underset{\sim}{f}_\rho) = \sup \mathrm{Rang}(\rho)$" as if $\underset{\sim}{G} \subseteq \mathbb{Q}$ is generic over $\mathbf{V}$, for any $\alpha < \sup \mathrm{Rang}(\rho)$ for some $p \in \mathbf{G}$ we have $\alpha \in u^p \wedge \rho \in \Lambda^p$ by Claim 4.8, and there is q such that $p \leq q \in \mathbf{G}, p \neq q$ by Claim 4.9, so recalling (c) from the definition of $\leq_{\mathbb{Q}}$ we are done. Similarly $\Vdash_{\mathbb{Q}}$ "$\mathrm{Rang}(\underset{\sim}{f}_\rho) = \sup \mathrm{Rang}(\rho)$".

Also $\rho \triangleleft \varrho$ implies $\Vdash_{\mathbb{Q}} \underset{\sim}{f}_\rho \subseteq \underset{\sim}{f}_\varrho$. For the EF^+-version we have to analyze dependence relations, which is straightforward as in the proof of Theorem 5.2. So $\langle f_\rho : \rho \in \Lambda^* \rangle$ exemplifies the equivalence. Q.E.D.

Claim 4.17. $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{M}_1 = \mathrm{EM}_{\tau(T)}(\underset{\sim}{\mathbf{J}}_1, \Phi), \underset{\sim}{M}_2 = \mathrm{EM}_{\tau(T)}(\underset{\sim}{\mathbf{J}}_2, \Phi)$ are not isomorphic".

Proof. Let $\underset{\sim}{M}_\ell^+ = \mathrm{EM}(\underset{\sim}{\mathbf{J}}_\ell, \Phi)$ so $\underset{\sim}{M}_\ell^+ \restriction \tau(T) = \underset{\sim}{M}_\ell$ for $\ell = 1, 2$, and assume towards a contradiction that $p \in \mathbb{Q}$, and $p \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{g}$ is an isomorphism from $\underset{\sim}{M}_1$ onto $\underset{\sim}{M}_2$". For each $\delta \in S^{\lambda^+}_\lambda := \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda\}$ by Claims 4.2 and 4.3 we can find $p_\delta \in \mathbb{Q}$ above p and g_δ such that $p \leq p_\delta, \delta \in u^{p_\delta}$, $p_\delta \Vdash$ "g_δ is $\underset{\sim}{g} \restriction \mathrm{EM}(\mathbf{J}_1^{p_\delta}, \Phi)$", and g_δ is an isomorphism from $\mathrm{EM}_{\tau(T)}(\mathbf{J}_1^p, \Phi)$ onto $\mathrm{EM}_{\tau(T)}(\mathbf{J}_2^p, \Phi)$.

We can find stationary $S \subseteq S^{\lambda^+}_\lambda$ and p^* such that $p_\delta \restriction \delta$, defined in Claim 4.11 is p^* for $\delta \in S$, for $\delta_1, \delta_2 \in S$, $u^{p_{\delta_1}}, u^{p_{\delta_2}}$ has the same order type and the order preserving mapping π_{δ_1,δ_2} from $u^{p_{\delta_2}}$ onto $u^{p_{\delta_1}}$ induce an isomorphism from p_{δ_2} onto p_{δ_1}, and if $\delta_1 < \delta_2 \in S$ then $\sup(u^{p_{\delta_1}}) < \delta_2$.

Now choose $\eta^* = \langle \delta_n^* : n < \omega \rangle$ such that $\delta_n^* < \delta_{n+1}^*$, $\delta_n^* = \sup(S \cap \delta_n^*)$ and $\delta_n^* \in S$, and let $\delta^* = \sup\{\delta_n^* : n < \omega\}$.

We can now define $q \in \mathbb{Q}$ as follows $u^q = \bigcup\{p_{\delta_n^*} : n < \omega\}$, $<^q = \{(\alpha, \beta) : \alpha <^{p_{\delta_n^*}} \beta$ for some n or $\alpha + \lambda \leq \beta \wedge \{\alpha, \beta\} \subseteq u^q$, equivalently for some $m < n, \alpha \in u^{p_{\delta_m^*}} \setminus \delta_m^*$ and $\beta \in u^{p_{\delta_n^*}} \setminus \delta_n^*\}$, $\mathfrak{S}_1^q = \bigcup\{\mathfrak{S}_1^{p_{\delta_n^*}} : n < \omega\} \cup \{\eta^*\}$, $\mathfrak{S}_2^q = \bigcup\{\mathfrak{S}_2^{p_{\delta_n^*}} : n < \omega\}$, $\Lambda^q = \cup\{\Lambda^{p_{\delta_n^*}} : n < u\}$, and $f_\rho^q = f_\rho^{p_{\delta_n^*}}$ if $\rho \in \Lambda^{p_{\delta_n^*}}$.

So there is a pair (q_*, g^+) such that $q \leq_{\mathbb{Q}} q_*$, $q_* \Vdash_{\mathbb{Q}}$ "$g^+ = \underset{\sim}{g} \restriction \mathrm{EM}(\mathbf{J}_1^{q_*}, \Phi)$, and g^+ is an automorphism of $\mathrm{EM}_{\tau(T)}(\mathbf{J}^{q_*}, \Phi)$. So $g^+(a_{\eta^*}) \in \mathrm{EM}(\mathbf{J}_2^{q_*}, \Phi)$ hence is of the form $\sigma^{M_2^+}(a_{t_0}, \ldots, a_{t_{n-1}})$ for some $t_0, \ldots, t_{n-1} \in \mathbf{J}_2^{q_*}$ and a τ_Φ-term $\sigma(x_0, \ldots, x_{n-1})$. Note that by the definition of $\leq_{\mathbb{Q}}$:

Subclaim 4.18. If $\eta \in \mathfrak{S}_2^{q_*}$ then $\mathrm{Rang}(\eta) \cap u^q$ is bounded in δ^*.

Proof. If $\eta \in \mathfrak{S}_2^q$ this holds by our choice of q and if $\eta \in \mathfrak{S}_2^{q_*} \backslash \mathfrak{S}_2^q$ then $\mathrm{Rang}(\eta) \cap u^q$ is finite so as $u^q \subseteq \delta$ it follows that $\mathrm{Rang}(\eta) \cap u^q$ is bounded in δ^*. Q.E.D.

We can find $n(*) < \omega$ such that for each $k < n$ and $\ell < n$ we have

(a) if $t_\ell \in Q^{\mathbf{J}_2^{q_*}}$, i.e., $t_\ell \in u^{q_*} \subseteq \lambda^+$ then $t_\ell \leq^q \delta_{n(*)}^*$, and

(b) if $t_\ell \in P^{\mathbf{J}_2^{q_*}}$, i.e., $t_\ell \in \mathfrak{S}_2^{q_*}$ then $\{F_n^{\mathbf{J}_2^{q_*}}(t_\ell) : n < \omega\}$ is disjoint to $[\delta^*_{n(*)}, \delta^*) \cap u^q$,

using "T is stable". The rest of the proof is exactly as in Propositions 3.5 and 3.7. Q.E.D. (Claim 4.17)

Q.E.D. (Theorem 4.1)

5 Theories with order

Recall from [3, 3.19]:

Theorem 5.1. If $\lambda = \mu^+$, $\mathrm{cf}(\mu), \lambda = \lambda^{<\kappa}, \kappa = \mathrm{cf}(\kappa) < \kappa(T)$ and T is unstable, then there are $\mathrm{EF}_{\mu\times\kappa,\lambda^+}$-equivalent non-isomorphic models of T of cardinality λ.

The new point in the following Theorem 5.2 is the use of EF^+ rather than EF.

Theorem 5.2. Assume $\lambda = \lambda^{<\theta}$ and λ is regular uncountable, $T \subseteq T_1$ are complete first order theories of cardinality $< \lambda$.

1. If T is unstable then there are models M_1, M_2 of T_1 of cardinality λ^+, $\mathrm{EF}^+_{\lambda,\theta,\lambda^+}$-equivalent with non-isomorphic τ_T-reducts.

2. Assume $\Phi \in \Upsilon^{\mathrm{or}}_\kappa$ is proper for linear orders, $\bar\sigma = \langle \sigma_i(x) : i < i(*)\rangle$ a sequence of terms from $\tau_\Phi, \bar x^\ell = \langle x_i^\ell : i < i(*)\rangle, i(*) < \lambda, \varphi(\bar x^1, \bar x^2)$ is a formula in $\mathcal{L}(\tau_T), \tau \le \tau_T$ (any logic) and for every linear order I letting $M = \mathrm{EM}(I,\Phi), \bar b_t = \langle \sigma_i^M(a_t) : i < i(*)\rangle$ we have $(M{\restriction}\tau) \models \varphi[\bar b_s, \bar b_t]^{\mathrm{if}(s<t)}$ for every $s,t \in I$. Then there are linear orders I_1, I_2 of cardinality λ^+ such that M_1, M_2 are $\mathrm{EF}^+_{\lambda,\theta,\lambda^+}$-equivalent but not isomorphic where $M_\ell = \mathrm{EM}_\tau(I_\ell, \Phi)$ for $\ell = 1,2$.

3. If every $\mathrm{EM}_\tau(I,\Phi)$ is a model of T_1 then in 2., the models M_1, M_2 are in $\mathrm{PC}(T_1, T)$.

Proof. To see 1., let $\varphi(\bar x, \bar y) \in \mathcal{L}(\tau_T)$ order some infinite subset of ${}^mM$ for some $M \models T$. Let Φ be as in Definition 3.3, i.e., proper for linear orders such that $\tau_{T_1} \subseteq \tau(\Phi), |\tau(\Phi)| = |T_1|$ and for every linear order I,$\mathrm{EM}(I,\Phi)$ (we allow the skeleton to consist of m-tuples rather than elements) is a model of T_1 satisfying $\varphi[\bar a_s, \bar a_t]$ iff $s <_I t$. Now we can apply part 2. with $i(*) = m$.

In order to prove 2., we choose I such that I is a linear order of cardinality λ (yes, not λ^+), if $\alpha, \beta \in (1,\lambda]$ then $(I\times\alpha) + (I\times\beta)^* \cong I$ (equivalently every $\alpha,\beta \in [1,\lambda^+)$), I is isomorphic to its inverse, and I has cofinality λ. For every $S \subseteq S^{\lambda^+}_\lambda = \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda\}$ we define $I_S = \sum_{\alpha<\lambda^+} I_{S,\alpha}$

where $I_{S,\alpha}$ is isomorphic to I if $\alpha \in \lambda^+ \backslash S$ and isomorphic to the inverse of $I \times \omega$ otherwise.

Claim 5.3. If $S_1, S_2 \subseteq S^{\lambda^+}_\lambda$ then the models $\mathrm{EM}(I_{S_2}, \Phi)$, $\mathrm{EM}(I_{S_1}, \Phi)$ are $\mathrm{EF}^+_{\lambda,\theta,\lambda^+}$-equivalent.

Proof. Let $J_{\ell,\gamma} = \sum_{\alpha<\gamma} I_{S_\ell,\alpha}$. Let $\mathcal{F} := \{f$: for some non-zero ordinal $\gamma < \lambda^+, f \in \mathcal{F}_\gamma$ and $[\gamma \in S_1 \Leftrightarrow \gamma \in S_2]\}$ where $\mathcal{F}_\gamma := \{f$ is an isomorphism from $\sum_{\alpha<\gamma} I_{S_1,\alpha}$ onto $\sum_{\alpha<\gamma} I_{S_2,\alpha}\}$. Then we have:

- $\mathcal{F}_\gamma \neq \varnothing$ for $\gamma < \lambda^+$.
- If $f \in \mathcal{F}_\gamma$ and $[\gamma \in S_1 \equiv \gamma \in S_2]$ and $\gamma < \beta < \lambda$ then f can be extended to some $g \in \mathcal{F}_\beta\}$.
- If $\gamma < \lambda, X_\ell \subseteq I_\ell$ has cardinality $< \lambda^+$ for $\ell = 1, 2$ then for some successor $\beta, \gamma < \beta < \lambda^+$ and $X_\ell \subseteq J_{\ell,\beta}$ for $\ell = 1, 2$.
- If $\gamma_i \in S_1 \Leftrightarrow \gamma_i \in S_2$ for $i < \delta, \delta$ a limit ordinal $< \lambda$ and $\langle \gamma_i : i < \delta \rangle$ is increasing then $\gamma_\delta := \bigcup\{\gamma_i : i < \delta\}$ satisfies $\gamma_\delta \in S_1 \equiv \gamma_\delta \in S_1$.

Lastly, we have to deal with **Case 2** in Definition 2.5, so let us assume that $f_* \in \mathcal{F}_{\gamma_*}, [\gamma_* \in S_1 \equiv \gamma_* \in S_2]$ and $\mathbf{R}_\ell \subseteq {}^{\theta>}(M_\ell)$ for $\ell = 1, 2$ are as there for f_*. This holds because the strategy is simple, e.g., with no memory. Now if f does not map the definition of $\mathbf{R}_1$ in M_1 to the definition of $\mathbf{R}_2$ in M_2 we can use **Subcase 2B** there, so we assume this does not occur. Let $\ell \in \{1, 2\}$, and get:

- Let $\mathbf{e}_\ell = \{(\bar{s}, \bar{t}) : \bar{s}, \bar{t} \in {}^{\theta>}(I_\ell)$ and some automorphism of I_ℓ over I_{ℓ,γ_*} maps $\bar{s}$ to $\bar{t}\}$.
- Let Y_ℓ be the set of $\mathbf{e}_\ell$-equivalence classes.

Note that for $\ell \in \{1, 2\}, n < \omega$ and $\mathbf{y}_0, \dots, \mathbf{y}_n \in Y_\ell$ the following are equivalent:

(a) some $\bar{a} \in \mathbf{y}_n$ depend (by $\mathbf{R}_1$) on $\mathbf{y}_0 \cup \ldots \cup \mathbf{y}_{n-1}$, and

(b) every $\bar{a} \in \mathbf{y}_n$ depends (by $\mathbf{R}_1$) on $\mathbf{y}_0 \cup \ldots \cup \mathbf{y}_{n-1}$.

So $\mathbf{R}_1$ induce a 1-dependence relation on Y_1, so let $\langle \mathbf{y}_i : i < i(*) \rangle$ be a maximal independent subset of Y_1. Therefore, it is enough to deal with one $\mathbf{y}_i$. Now we can find $\bar{t}_{i,\gamma} \in \mathbf{y}_i$ such that $\mathrm{Rang}(\bar{t}_{i,\gamma}) \backslash I_{\ell,\gamma_*} \subseteq I_{\ell,\gamma+2} \backslash I_{\ell,\gamma+1}$ for each $\gamma \in [\gamma_*, \lambda^+)$ as I_1 has enough automorphisms. If $\{\bar{t}_{i,\gamma} : \gamma \in [\gamma_*, \lambda^+)\}$ is not $\mathbf{R}_1$-independent, then $\dim(\mathbf{y}_i)$ is finite, in fact 1 or 0. So we choose β_* such that $\gamma_* < \beta_* < \lambda^+$ and $\beta_* \in S_1 \equiv \beta_* \in S_2$ and for every $i < i(*)$, if $\dim(X_{\mathbf{y}_i})$ is finite then $\mathbf{y}_i$ has a maximal $\mathbf{R}_1$-independent set included in ${}^{\varepsilon(\mathbf{y}_i)}(J_{1,\beta_*})$. [Why is this possible? Because for any such β_* is an automorphism of I_2 over J_{1,γ_*} mapping I_{β_*+2} onto I_{γ_*+2}.]

Let $g \in \mathcal{F}_{\beta_*}$ extend f; using it we can choose $\langle (\bar{a}^1_\zeta, \bar{a}^2_\zeta) : \zeta < \zeta^* \rangle$ as required. Q.E.D. (Claim 5.3)

Claim 5.4. If $S_1, S_2 \subseteq S^{\lambda^+}_\lambda$ and $S_1 \backslash S_2$ is stationary, then $\mathrm{EM}_\tau(I_{S_1}, \Phi)$, $\mathrm{EM}_\tau(I_{S_2}, \Phi)$ are not isomorphic.

Proof. This is similar to the proof in [10, III,§3] (or [15, III,§3] = [16, §3]), only easier. In fact, imitating it we can represent the invariants from there. Q.E.D.

Part 3. of Theorem 5.2 is obvious. Q.E.D. (Theorem 5.2)

Corollary 5.5. Assume T is a (first order complete) theory.

1. If T is unstable, then $(T, *)$ is fat.
2. If T is unstable or stable with DOP, or stable with OTOP, then T is fat.
3. For every μ there is a μ-complete, class forcing $\mathbb{P}$ such that in $\mathbf{V}^{\mathbb{P}}$ we have: if T is not strongly dependent or just not strongly stable, then T is fat, moreover $(T, *)$ is fat.

Proof. We get 1. by Theorem 5.2. The proof of 2. is similar,the only difference is that the formula defining the "order" is not first order and the length of the relevant sequences may be infinite but still $\le |T|$ (cf. [11, XIII]).

Now, by 1. and 2., we should consider only stable, not strongly stable T. Choose a class $\mathbf{C}$ of regular cardinals such that $\lambda \in \mathbf{C} \Rightarrow (2^{<\lambda})^+ < \mathrm{Min}(\mathbf{C} \backslash \lambda^+)$ and $\mathrm{Min}(\mathbf{C}) > \mu$. We iterate with full support $\langle \mathbb{P}_\mu, \underset{\sim}{\mathbb{Q}}_\mu : \mu \in \mathbf{C} \rangle$ with $\underset{\sim}{\mathbb{Q}}_\mu$ as in Theorem 4.1. Q.E.D.

Proposition 5.6. Assume $T \subseteq T_1, \lambda = \lambda^\kappa$ is not necessary regular and $\kappa = \mathrm{cf}(\kappa) < \kappa(T)$, e.g. T is unstable. Then there are $\mathrm{EF}^+_{\lambda\times\kappa,\lambda,\lambda^+}$-equivalent non-isomorphic models from $\mathrm{PC}(T_1, T)$ of cardinality λ^+.

Proof. As in [3], following the proof of Theorem 5.2. Q.E.D.

We get the following corollaries via old results, as mentioned in the introduction. Note that Corollary 5.7 is on elementary classes and Corollary 5.8 on small enough pseudo elementary classes.

Corollary 5.7 (ZFC)**.** For first order countable complete first order theory T the following conditions are equivalent:

(A) The theory T is superstable with NDOP and NOTOP.

(B)$_1$ If $\lambda = \mathrm{cf}(\lambda) > |T|$ and $M_1, M_2 \in \mathrm{Mod}_T(\lambda)$ are $\mathcal{L}_{\infty,\lambda}(\tau_T)$-equivalent then M_1, M_2 are isomorphic.

(B)$_2$ Like (B)$_1$ for some $\lambda = \mathrm{cf}(\lambda) > |T|$.

(C) If $\lambda = \mathrm{cf}(\lambda) > |T|$ and $M_1, M_2 \in \mathrm{Mod}_T(\lambda^+)$ are $\mathrm{EF}_{\omega,\lambda}$-equivalent then M_1, M_2 are isomorphic.

(D) For some regular $\lambda > |T|$, if $M_1, M_2 \in \mathrm{Mod}_T(\lambda^+)$ are $\mathrm{EF}_{\lambda,\lambda^+}$-equivalent then they are isomorphic.

Proof. Clauses (A), (B)$_1$, and (B)$_2$ are equivalent: in [11, XIII,Th.1.11], we proved (A) $\Rightarrow$ (B)$_1$ $\wedge$ (B)$_2$, and the other implication holds by [9]. Now by the definitions trivially (B)$_1$ $\Rightarrow$ (C) $\Rightarrow$ (D). Lastly, by Theorem 5.1 we have $\neg$(A) $\Rightarrow$ $\neg$(D), i.e., (D) $\Rightarrow$ (A) so we have the circle. Q.E.D.

Corollary 5.7 tells us what we know about (A)$_0$ and (B)$_0$ of Question 2.1.

Corollary 5.8 (ZFC)**.** For first order countable complete first order theory T and $\kappa \geq 2^{\aleph_0}$ the following conditions are equivalent:

(A) T is unsuperstable,

(B)$_\kappa$ for every $\lambda > \kappa \geq |T|$ and (κ, T)-candidate ψ (cf. Definition 1.2), and ordinal $\alpha < \lambda$ satisfying $|\alpha|^+ = \lambda \Rightarrow |\alpha \leq |\alpha| \times \omega$, there are $\mathrm{EF}_{\alpha,\lambda}$-equivalent non-isomorphic models $M_1, M_2 \in \mathrm{PC}_{\tau(T)}(\psi)$ of cardinality λ,

(C)$_\kappa$ for some $\lambda > \kappa \geq |T|$, for no (κ, T)-candidate ψ is the class $\mathrm{PC}_{\tau(T)}(\psi)$ categorical in λ.

Proof. First, assume T is superstable, so clause (A) holds. By the proofs of [11, VI,§4] there is a (κ, T)-candidate ψ such that $\mathrm{PC}_{\tau(T)}(\psi)$ is the class of saturated models of T, (in details, if $n < \omega, \bar{a} \in {}^n\mathfrak{C}$, $\mathrm{tp}(b, \bar{a}, \mathfrak{C})$ is stationary, $q = \mathrm{tp}(\bar{b}, \varnothing, \mathfrak{C}), p = p(x, \bar{y}) = \mathrm{tp}(\langle a\rangle^\frown \bar{b}, \varnothing, \mathfrak{C})$ then let $\psi_{p,q}$ be such that $M \models \psi_{p,q}$ iff for every $\bar{b}' \in {}^nM$ realizing the type $q(\bar{y})$, the function $c \mapsto F^M_{p,q}(c, \bar{b}')$ is one-to-one and if $k < \omega, c_0, \ldots, c_k \in M$ are pairwise distinct then $\mathrm{tp}_{\mathcal{L}(\tau(T))}(F^M_{p,q}(c_k), \{F^M_{p,q}(c_0), \ldots, F^M_{p,q}(c_{k-1})\} \cup \bar{b}', M)$ extends $p(x, \bar{b}')$ and does not fork over M. Lastly, $\psi = \bigwedge\{\psi_{p,q} : p, q \text{ as above}\}$ so $\in \mathcal{L}_{\kappa^+,\omega}$. So in the present case also (B)$_\kappa$ and (C)$_\kappa$ hold.

Secondly, assume T is not superstable, so clause (A) does not hold and we shall prove the rest. Let ψ be a (κ, T)-candidate. Take $\Phi \in \Upsilon^{\omega_1\text{-tr}}_\kappa$ witnessing unsuperstability and use Claim 1.3 and Theorem 5.1. Q.E.D.

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