Commutativity associated by Heun's differential equations

Mehmet Emir Koksal

Department of Mathematics, Ondokuz Mayis University, 55139 Samsun, Turkey
E-mail: mekoksal@omu.edu.tr

Abstract

In this study, commutativity of Heun's differential equation is considered. It is shown that the system modeled by Heun's differential equation has a commutative pair with (or without) some conditions or not. These conditions are presented explicitly. Theoretical results are supported by numerical experiments.

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1 Introduction

The commutativity is very important subject concerning engineers and mathematicians. Especially, it places an important role to improve different system performances in system and control design as a main branch of electrical and electronics engineering. For example, they are used in cascade connected and feedback systems to design high-order composite systems for achieving several beneficial properties such as controllability, sensitivity, robustness, and design flexibility.

Even though there is a large cycle of works on the commutativity of continuous time-varying linear system, there are only a few scientists studying on this subject. For the first time, this subject was studied in the literature by E. Marshall in 1982 [1]. His study was valid for the first-order linear time-varying differential systems. After that, commutativity conditions of second-order [2], third-order [3] and fourth-order [4] systems were obtained by M. Koksal in 1980s. After a long time, commutativity of fifth-order systems [5] was studied by M. Koksal and M.E. Koksal in 2011.

Depending on the revealing of the theoretical results on the commutativity of law-order continuous time-varying linear systems, applied problems started to be studies towards the end of the 1980s. Effects of commutativity over system performance were considered by M. Koksal in [6, 7, 8] in 1987, 1988, 1989. Over 30 years later, it is applied to develop an alternative method in cryptology by M. Koksal in [9].

Even though there are many literatures on the theory and applications of the commutativity of continuous time-varying linear systems, there are only a few works on the commutativity of discrete (digital) time-varying linear systems. Commutativity of second-order [10] and first-order [11] discrete-time linear time-varying systems were studied in [2015] and [2019] respectively.

Although most of the famous second-order linear time-varying differential systems were subject from the commutativity point of view in [12, 13], Heun's differential equation [14] is not among them. So, the purpose of this paper is to study commutativity conditions of Heun's differential equation; namely

$$\ddot{y} + \left(\frac{\gamma}{t} + \frac{\delta}{t-1} + \frac{\varepsilon}{t-a}\right)\, \dot{y} + \left[\frac{\alpha\beta t - b}{t(t-1)(t-a)}\right] y = 0, \tag{1.1}$$

where $\alpha+\beta+1 = \gamma+\delta+\varepsilon$. Here γ, δ, ε, α, β, a, b are constants. Note that $\ddot{y}_A(t) = y_A^{''}(t) = \frac{d^2}{dt^2} y_A\,(t)$, $\dot{y}_A(t) = y_A^{'}(t) = \frac{d}{dt} y_A\,(t)$ and similar notations will be followed throughout the paper.

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2 Commutativity conditions

Let A be second-order linear time-varying differential system described by

$$A : a_2(t)\ddot{y}_A(t) + a_1(t)\dot{y}_A(t) + a_0(t)y_A(t) = x_A(t); t \geq 0, \tag{2.1}$$

where $x_A(t)$ is the independent excitation and $y_A(t)$ is the resulting response. We assume time variable (initial time) and $y_A(t_0)$, $y'_A(t_0)$ are the initial conditions. For the unique solution of Eq. (2.1) for $t \geq t_0$, it is sufficient that the excitation and the time-varying coefficients $a_2(t)$, $a_1(t)$, $a_0(t)$ be piece-wise continuous functions of time with $a_2(t) \not\equiv 0$ [11].

It is shown in [12] that all the commutative pairs of (2.1) are obtained by

$$\begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} a_2 & 0 & 0 \\ a_1 & a_2^{0.5} & 0 \\ a_0 & \frac{a_2^{-0.5}(2a_1 - a_2')}{4} & 1 \end{bmatrix} \begin{bmatrix} k_2 \\ k_1 \\ k_0 \end{bmatrix}, \tag{2.2}$$

if the coefficients of A satisfy

$$-a_2^{0.5}\frac{d}{dt}\left[a_0 - \frac{1}{16a_2}\left(4a_1^2 + 3\left(a_2'\right)^2 - 8a_1a_2' + 8a_1'a_2 - 4a_2a_2''\right)\right]k_1 = 0.$$

In Eqs. (2.1) and (2.2), k_2, k_1, k_0 are some constants. Then any commutative pair B of A is described by

$$B : b_2(t)y_B''(t) + b_1(t)y_B'(t) + b_0(t)y_B(t) = x_B(t)$$

with initial conditions $y_B(t_0)$ and $y'_B(t_0)$. If $k_1 = 0$, Eq. (2.2) is automatically satisfied for any second-order linear time-varying differential system A. Since Eq. (2.2) is automatically satisfied for $k_1 = 0$, every second-order linear time-varying systems has a commutative pair which is its constant feedback conjugate. In this presentation, we look for the commutativity not of this type, so $k_1 \neq 0$, and the coefficients of A must satisfy

$$a_0 - \frac{1}{16a_2}\left[4a_1^2 + 3\left(a_2'\right)^2 - 8a_1a_2' + 8a_1'a_2 - 4a_2a_2''\right] = K \tag{2.3}$$

for all $t \geq t_0$ where K is some constant.

Differential systems A and B be commutative under nonzero initial conditions as well and their cascade connections AB and BA are described by a fourth-order linear time-varying differential system C described by

$$c_4(t)y^{(4)}(t) + c_3(t)y'''(t) + c_2(t)y''(t) + c_1(t)y'(t) + c_0(t)y(t) = x(t)$$

with initial conditions $y_A(t_0)$, $y'_A(t_0)$, $y''_A(t_0)$, $y'''_A(t_0)$.

3 Commutativity of Heun's differential equation

Under the light of the above given theoretical bases, we investigate the commutativity conditions for the Heun's differential equations given in (1.1).

With the coefficients of Heun's differential equation in (1.1)

$$a_2(x) = 1$$

$$a_1(x) = \frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\varepsilon}{x-a} \tag{3.1}$$

$$a_0(x) = \frac{\alpha\beta x - b}{x(x-1)(x-a)} \tag{3.2}$$

the commutativity condition in Eq. (2.3) yields

$$4a_0 - a_1^2 - 2a_1' = constant. \tag{3.3}$$

Putting Eqs. (3.1)-(3.2) and the formula $a_1'(x) = -\gamma x^-2 - \delta(x-1)^{-2} - \varepsilon(x-a)^{-2}$ in Eq. (3.3), we obtain the following equation:

$$\frac{4(\alpha\beta x - b)}{x(x-1))(x-a)} - \frac{\gamma^2}{x^2} - \frac{\delta^2}{(x-1)^2} - \frac{\varepsilon^2}{(x-a)^2} - \frac{2\gamma\delta}{x(x-1)} - \frac{2\gamma\varepsilon}{x(x-a)}$$
$$-\frac{2\delta\varepsilon}{(x-1)(x-a)} + \frac{2\gamma}{x^2} + \frac{2\delta}{(x-1)^2} + \frac{2\varepsilon}{(x-a)^2} = K.$$

Rearranging the above equation, we get

$$\frac{2\left(\alpha\beta - \gamma\delta - \gamma\varepsilon - \delta\varepsilon\right)x^2\left(x^2 - ax - x + a\right)}{x^2(x-1)^2(x-a)^2}$$
$$+\frac{2\left(-2b + \gamma\delta a + \gamma\varepsilon\right)x\left(x^2 - ax - x + a\right)}{x^2(x-1)^2(x-a)^2}$$
$$-\frac{\left(\gamma^2 - 2\gamma\right)\left(x^2 - 2x + 1\right)\left(a^2 - 2ax + x^2\right) + \left(\delta^2 - 2\delta\right)x^2\left(a^2 - 2ax + x^2\right)}{x^2(x-1)^2(x-a)^2}$$
$$-\frac{\left(\varepsilon^2 - 2\varepsilon\right)x^2\left(x^2 - 2x + 1\right)}{x^2(x-1)^2(x-a)^2} = K.$$

If we multiply the coefficient of the forces of x on the left side of the equation by multiplying the inside and the outside to the coefficient of the forces of x on the right side

For x^0,

$$-\left(\gamma^2 - 2\gamma\right)a^2 = 0 \tag{3.4}$$

For x^1,

$$2(-2b + \gamma\delta a + \gamma\varepsilon)a + 2\left(\gamma^2 - 2\gamma\right)a^2 + 2\left(\gamma^2 - 2\gamma\right)a = 0. \tag{3.5}$$

For x^2,

$$2(2\alpha\beta - \gamma\delta - \gamma\varepsilon - \delta\varepsilon)a + 2(-2b\gamma\delta a + \gamma\varepsilon)(-a-1)$$

$$-\left(\gamma^2 - 2\gamma\right)\left(a^2 + 4a + 1\right) - \left(\delta^2 - 2\delta\right)a^2 - \left(\varepsilon^2 - 2\varepsilon\right) = Ka^2 = 0. \tag{3.6}$$

For x^3,

$$-2(2\alpha\beta-\gamma\delta-\gamma\varepsilon-\delta\varepsilon)(a+1)+2(-2b+\gamma\delta a+\gamma\varepsilon)$$
$$+\left(\gamma^2-2\gamma\right)(2a+2)+\left(\delta^2-2\delta\right)+2a$$
$$+\left(\varepsilon^2-2\varepsilon\right)+2=K\left(-2a^2-2a\right)=0. \tag{3.7}$$

For x^4,

$$2\left(2\alpha\beta-\gamma\delta-\gamma\varepsilon-\delta\varepsilon\right)-\left(\gamma^2-2\gamma\right)$$
$$-\left(\delta^2-2\delta\right)-\left(\varepsilon^2-2\varepsilon\right)=K\left(a^2+4a+1\right)=0. \tag{3.8}$$

For x^5,

$$(-2a-2)K=0. \tag{3.9}$$

For x^6,

$$K=0. \tag{3.10}$$

Although the set of nonlinear algebraic equations (3.4)-(3.10) and $\alpha+\beta+1=\gamma+\delta+\varepsilon$ in Eq. (1.1) seem to be complicated to determine the proper values and/or relations of the parameters a, b, δ, γ, ε, α, β, a systematic process of simplification yields the results in Table 1.

Table 1

$a=0$	$b=\frac{1}{4}(\gamma+\varepsilon)(\gamma+\varepsilon-2)$	$\delta=0$	$\alpha=\frac{1}{2}(\gamma+\delta+\varepsilon)$
		$\delta=2$	$\beta=a-1$
$a=1$	$\gamma=0$	$b=0$	*or*
	$\gamma=2$	$b=\delta+\varepsilon$	*vice versa*

4 Examples

Example 1: In accordance to the results of Table 1, considering Eq. (1.1) for $\alpha=\delta=0$, $\beta=\frac{\gamma+\varepsilon}{2}$, $\alpha=\frac{\gamma+\varepsilon}{2}-1$, we obtain

$$\ddot{y}+\frac{\varepsilon+\gamma}{x}\dot{y}+\frac{(\gamma+\varepsilon)(\gamma+\varepsilon-2)(x-1)}{4x^2\,(x-1)}y=0 \tag{4.1}$$

In the above equation, $a_2(x)=1$, $a_1(x)=\frac{\gamma+\varepsilon}{x}$, $a_0(x)=\frac{(x-1)(\gamma+\varepsilon)(\gamma+\varepsilon-2)}{4x^2(x-1)}$. For the existence of the commutative pair of the system modeled by Eq. (4.1), Eq. (3.3) should be satisfied and it is equal to 0 for this example. Using Eq. (2.2), we find the coefficients of commutative pair of the system A as the follows:

$$b_2=k_2,$$

$$b_1=\frac{(\gamma+\varepsilon)}{x}k_2+k_1,$$

$$b_0=\frac{(\varepsilon+\gamma)(\varepsilon+\gamma-2)(x-1)}{4x^2\,(x-1)}k_2+\frac{(\varepsilon+\gamma)}{2x}k_1+k_0.$$

So, system B is modelled by the following differential equation:

$$k_2\ddot{y}_B(t) + \dot{y}_B(t)\left[\frac{(\gamma+\varepsilon)}{x}k_2 + k_1\right]$$

$$+\left[\frac{(\varepsilon+\gamma)(\varepsilon+\gamma-2)(x-1)}{4x^2\,(x-1)}k_2 + \frac{(\varepsilon+\gamma)}{2x}k_1 + k_0\right]y_B(t) = 0.$$

For $\varepsilon = 2$, $\gamma = -1$, $k_2 = 3$, $k_1 = -3$, $k_2 = 0$, the above equations result with the commutative pairs

$$A : y_A'' + \frac{1}{x}y_A' - \frac{1}{4x^2}y_A = 0.$$

$$B : y_B'' + \left(\frac{1.5}{x} - 3\right)y_B' - \frac{6x+3}{4x^2}y_B = 0.$$

The cascade connections AB and BA are exicited by a pulse of amplitude 1000 and duration 0.1. The responses are found by Simulink using fixed step size of 0.001 and solver ode8 (Dormant-Prince). The results are plotted in Fig. 1 that also shows the input pulse. It is obviously true that the responses of both cascade connections AB and BA are identical, which supports the commutativity of A and B. Note that the commutative pair B of A is not a Heun's differential equation due to the constant term -3 in the second coefficient.

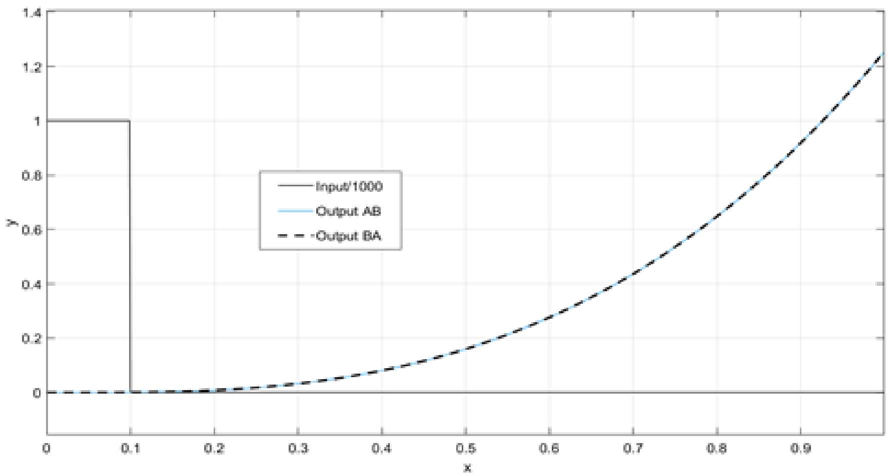

FIGURE 1. Simulation results for Example 1

Example 2: For the choice $a = 0$, $\gamma = \varepsilon = 0$, $\delta = 2$, Table 1 yields $b = 0$, $\alpha = 1$, $\beta = 0$. For these values of the parameters, Heun differential equation in (1.1) reduces to

$$y'' + \frac{2}{x-1}y' = 0. \tag{4.2}$$

In the above equation, $a_2(x) = 1$, $a_1(x) = \frac{2}{x-1}$, $a_0(x) = 0$. For the commutativity of the system modeled by Eq. (4.2) with another second-order linear system with varying coefficients, Eq. (3.3) should be satisfied and it is equal to 0 for this example. Using Eq. (2.2), we find the coefficients of the commutative pair of the system A as the follows:

$$b_2 = k_2,$$
$$b_1 = \frac{2}{x-1}k_2 + k_1,$$
$$b_0 = k_2 + \frac{1}{x-1}k_1 + k_0.$$

So, system B is modelled by the following differential equation:

$$k_2 y_B''(t) + \left(\frac{2}{x-1}k_2 + k_1\right) y_B'(t) + \left(k_2 + \frac{1}{x-1}k_1 + k_0\right) y_B(t) = 0.$$

Note that none of the commutative pairs of A are of Heun type due to the nonzero values of the second and third coefficients as $x \to \infty$. The subsystem B for $k_2 = 1$, $k_1 = 4$, $k_0 = 9$. Simulations of the cascade connected systems AB and BA are performed. The results are shown in Fig. 2. A periodic sequence of pulse train with amplitude 1000, period 4, bias -100, and duty cycle of 10% is applied starting at $x = 2$ until $x = 16$. Automatic solver selector is used with a fixed step-size of 0.01. It is seen that both of the connections AB and BA give the same step response.

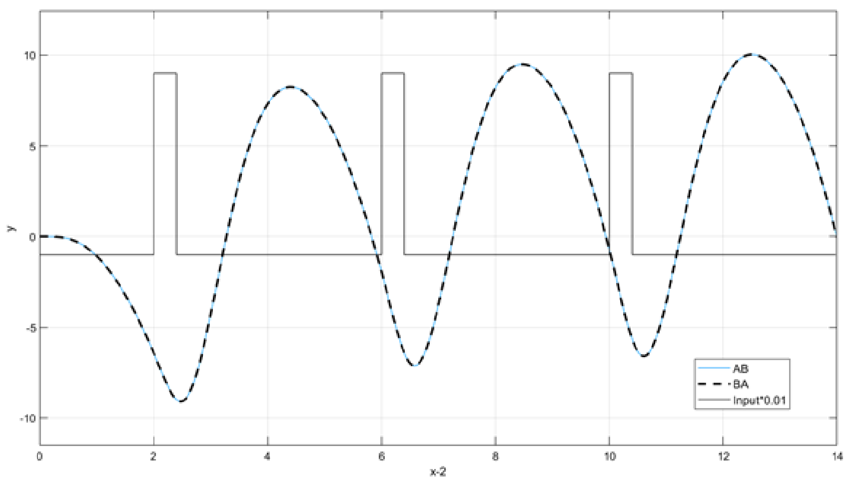

FIGURE 2. Simulation results for Example 2

Example 3: In accordance to the results of Table 1, considering Eq. (1.1) for $\gamma = \delta = 2$, $a = 1$, $\beta = \frac{\varepsilon}{2} + 1$, $\alpha = \frac{\varepsilon}{2} + 2$, $b = \varepsilon + 2$, we obtain

$$A : y'' + \frac{x(4+\varepsilon) - 2}{x(x-1)}y' + \frac{(\varepsilon+2)[x(\varepsilon+4) - 4]}{4x(x-1)^2}y = 0. \tag{4.3}$$

In the above equation, $a_2(x) = 1$, $a_1(x) = \frac{x(4+\varepsilon)-2}{x(x-1)}$, $a_0(x) = \frac{(\varepsilon+2)[x(\varepsilon+4)-4]}{4x(x-1)^2}$. For the commutativity of the system modeled by Eq. (4.3), Eq. (3.3) should be satisfied and the constant is equal to 0 for this example. Using Eq. (2.2), we find the coefficients of commutative pair of the system A as the follows:

$$b_2 = k_2,$$

$$b_1 = \frac{x(4+\varepsilon)-2}{x(x-1)}k_2 + k_1,$$

$$b_0 = \frac{(\varepsilon+2)\left[x(\varepsilon+4)-4\right]}{4x(x-1)^2}k_2 + \frac{x(4+\varepsilon)-2}{x(x-1)}k_1 + k_0.$$

So, system B is modelled by the following differential equation:

$$y_B''(t) + \left[\frac{x(4+\varepsilon)-2}{x(x-1)}k_2 + k_1\right] y_B'(t)$$

$$+k_2 \left\{ \frac{(\varepsilon+2)\left[x(\varepsilon+4)-4\right]}{4x(x-1)^2}k_2 + \frac{x(4+\varepsilon)-2}{x(x-1)}k_1 + k_0 \right\} y_B(t) = 0.$$

For simulations, the choice $\varepsilon = 2$, $k_2 = 1$, $k_1 = -1$, $k_0 = 1$ yields the commutative pairs

$$A : y'' + \frac{6x-2}{x(x-1)}y' + \frac{(6x-4)}{x(x-1)^2}y = 0,$$

$$B : y'' + \frac{-x^2+7x-2}{x(x-1)}y' + \frac{(x^3-5x^2+11x-5)}{x(x-1)^2}y = 0.$$

A composite input (namely, a repeating sequence with both time and output values 0 and 2 plus a step with step time 4, initial and final values −4 and 0) is used in the simulation. The input is shown in Fig. 3. The responses of the connections AB and BA are also shown on the same figure. Obviously they are identical, which proves the commutativity.

5 Conclusion

Heun's differential equation is examined in this study from the point of view commutativity. It is shown that a system described by Heun's differential equation has always a second order commutative pair with a proper choice of its parameters. The case of nonzero case of initial conditions is not examined in this study. The results are well validated by the MATLAB simulation software Simulink.

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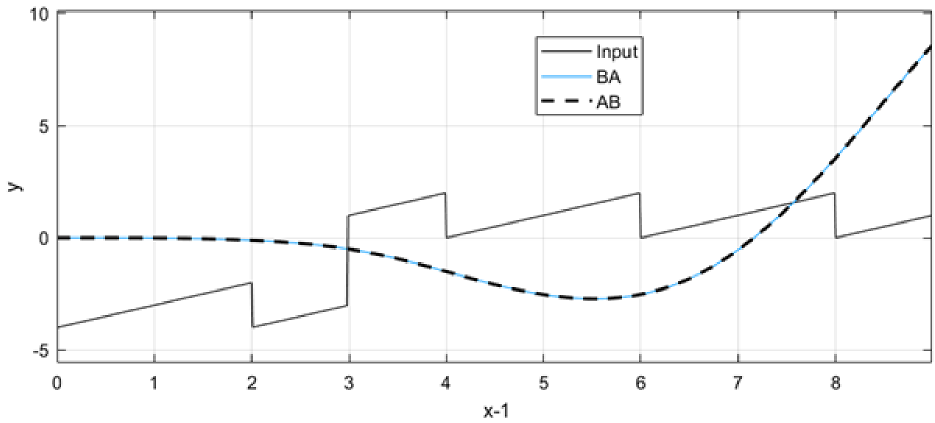

FIGURE 3. Simulation results for Example 3

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